Analogue Transformation Acoustics

An alternative theoretical framework for acoustic metamaterials

Final Report

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Transformation acoustics and analogue gravity

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Contents

1 Introduction

Transformational approaches have revolutionized the way in which metamaterials are designed. Using the symmetries of the fundamental equations that govern the physics of a certain class of materials, one is able to construct devices with highly non-trivial properties.

In the last few years, transformational approaches have been applied with great success in optics. Transformation optics is an emerging field (first proposed in 2006) [1, 2] that enables us to create curved geometries for light by suitably changing the optical medium it propagates through [3]. Usually, the electromagnetic properties (constitutive parameters) required for the implementation of the above-mentioned curved geometries cannot be found in nature. However, the recent development of metamaterials facilitates the synthesis of such exotic media. Thus, with the aid of metamaterials, transformation optics provides a new arena for creating optical devices that would be extremely difficult to achieve with classical design methods.

There already exist several interesting theoretical proposals and experimental work in this branch of research. Among the most striking proposals we can find cloaking devices [1, 2], electromagnetic wormholes [4], or black holes [5], to name a few. Transformation optics even provides a geometrical interpretation of the negative-index superlens [5, 6], as well as of a positive-index alternative [7]. Other devices based on transformation optics include special lenses [6], beam shifters and splitters [8], polarization rotators [9], superscatterers [10], or illusion-optics devices [11]. The interested reader can consult additional references from the bibliography section [3, 12, 13, 14].

More recently there has been an effort to apply transformational approaches to acoustic systems. Thus, following the procedure of transformation optics, a correspondence between material constitutive parameters and coordinate transformations has been applied in acoustics. Transformation acoustics is based on the invariance of the acoustic wave equation under a particular set of coordinate transformations [15, 16]. The more successful application of transformation acoustics has been acoustic cloaking, where acoustic metamaterials have been designed to guide sound waves around objects in two and three dimensions [17, 18]. Another prediction of transformation acoustics includes "acoustic external cloaks", where the concealed object and the sources are located in the same domain and the object is exposed to external sound. Ground cloaks is also an outcome of transformation acoustics that has been experimentally proved by using layers of perforated panels to obtain the material parameters needed for its functionality [19].

Unfortunately the acoustic equations are structurally very different from the Maxwell equations. In particular, the acoustic equations are not covariant and cannot be extended to curved spacetime as easily as the Maxwell equations. In addition, the relation between the acoustic properties of a medium and curved spacetime is not as direct as in the case of electromagnetic materials.

These issues are behind the slower progress in the development of a transformational acoustics as powerful as transformational optics and motivate the search for new and more powerful methods to design acoustic matamterials.

In this report we present a new approach to the problem of transformational acoustics which makes use of some fundamental insights from analogue gravity. The main idea is to design a metamaterial not via the exploitation of the direct symmetries of the acoustic equations, but recognizing the connection between these equations and the evolution of a scalar field in a (curved) spacetime which encodes all the physical information of the acoustic equations. In this way, the symmetries of this spacetime correspond to symmetries in the equations and the transformation approach can exploit this analogy. In the case of sound propagating in fluids, the analogue gravity equations are coincident with the equations for the scalar acoustic potential ϕ, and the symmetries of the abstract spacetime mirror the ones of the equations for this potential. In itself, the use of the equations for ϕ instead of the ones for the isotropic pressure offers already many advantages which, to our knowledge, were never recognized before. The use of analogue gravity as a framework to design the desired dynamics of ϕ, however, greatly simplifies the treatment of complicated situations (like the case of non-zero velocity of the fluid) and allows an easier implementation of typical general relativistic systems like black holes, trapped surfaces, etc.

Our specific study is limited to the study of sound propagation in (meta)fluids, but this is not as strong a limitation as it could appear. In fact in many instances acoustic metamaterials are modeled as effective fluids, precisely due to the use of fluids to induce the desired variation of density/sound speed [20]. Although this is not part of the present work, we expect that our approach could also be extended to elasticity, and hope to develop this idea (and some other related ones) in the near future.

The report is organized as follows. We start with a brief review of transformational optics in section 2. In section 3, we present the standard approach to transformational acoustics and point out its limitations. The acoustic equations are derived in terms of the scalar potential in section 4, and this is linked to the fundamentals of analogue gravity which are sketched in section 5. In section 6 we introduce the idea of Analogue Transformational Acoustics and in section 7 we give some examples of its applications. Section 8 contains several numerical examples illustrating the results of the previous sections. Finally, in section 9 we offer some conclusions and draw perspectives for future development.

2 Transformation optics

The concept of transformation optics is based on two basic observations. The first is the fact that light rays follow geodesics and these geodesics are distorted when spacetime has a non-zero curvature. The second is that light rays[1] are distorted when light propagates in a medium. In this sense the curved spacetime can be considered as an effective medium.

[1] In fact, this way of imagining the situation, although useful, is not totally precise. Light paths or light rays are the product of the geometric optics approximation and are to be considered only a limited representation of the real behavior of electromagnetic waves. However, for our purposes, such visualization can help to clarify the idea behind transformational optics.

In transformation optics this principle is used to design metamaterials in which light behaves in an exotic way. The application of transformation optics is based on the form invariance of Maxwell's equations under coordinate transformations and it is sketched in Fig. 1 for the simple case of purely spatial transformations[2].

We start from Maxwell's equations in flat empty space (which is usually called "virtual space"). Using a three dimensional metric γ_{ij} it is easy to express these equations in an arbitrary spatial coordinate system x^i, with $i = 1, 2, 3$ (Step 1). The solutions in this system are well known and can be visualized as a bundle of "light paths" that fills the space(-time). At this point, by means of a spatial coordinate transformation $f : x^i \to \bar{x}^{\bar{i}} = \Lambda^{\bar{i}}_k x^k$ we can distort these light paths in any (continuous) chosen way. Because of this covariance, Maxwell's equations can be written in the new coordinate system simply by using a spatial metric $\bar{\gamma}_{ij}$ (Step 2).

Since we have just performed a change of coordinates, there is, of course, no physical difference between Step 1 and Step 2. However, using the fact that there is no formal difference between the distortion of light rays due to the coordinate transformation and the presence of a certain medium, we can require that the solution that generates our distorted bundle has to belong to a set of Maxwell's equations in flat space (the "physical space") and in the presence of a certain anisotropic medium characterized by a permittivity ε^{ij} and a permeability μ^{ij} (Step 3). If we make $\tilde{x} = \bar{x}$ and require that the coordinate system in Step 1 and Step 3 is the same (reinterpreting $\tilde{x}$ as representing the same coordinates as x), then the properties of the material are determined as

$$\rho_r = \pm\frac{\sqrt{\gamma}}{\sqrt{\bar{\gamma}}}\rho \qquad J^i = \pm\frac{\sqrt{\gamma}}{\sqrt{\bar{\gamma}}}j^i \tag{1}$$

$$\varepsilon^{ij} = \mu^{ij} = \pm\frac{\sqrt{\gamma}}{\sqrt{\bar{\gamma}}}\bar{\gamma}^{ij} \tag{2}$$

Note that the $\pm$ sign appearing in the equations of Fig. 1 should be a $+$ sign for right-handed systems and a $-$ sign for left-handed systems. Finally, it is worth mentioning that the components of the fields in Step 3 must be the same as those in Step 2, so that the fields in Step 1 are related to the ones in Step 3 by the following rescaling

$$\tilde{E}_i = \bar{E}_i \qquad \bar{E}_{\bar{i}} = \Lambda^i_{\bar{i}} E_i, \tag{3}$$

$$\tilde{H}_i = \bar{H}_i \qquad \bar{H}_{\bar{i}} = \Lambda^i_{\bar{i}} H_i. \tag{4}$$

3 Standard Transformation Acoustics (STA)

In this section we will summarize briefly the classical approach to transformation acoustics. Our summary is not, by any means, complete and has the only purpose of offering a comparison ground with our new approach to transformation acoustics. We will begin

[2]The theory can, of course, be formulated for general space-time transformations. We have restricted ourselves to this case for the sake of simplicity and to allow an easier comparison with the case of acoustics.

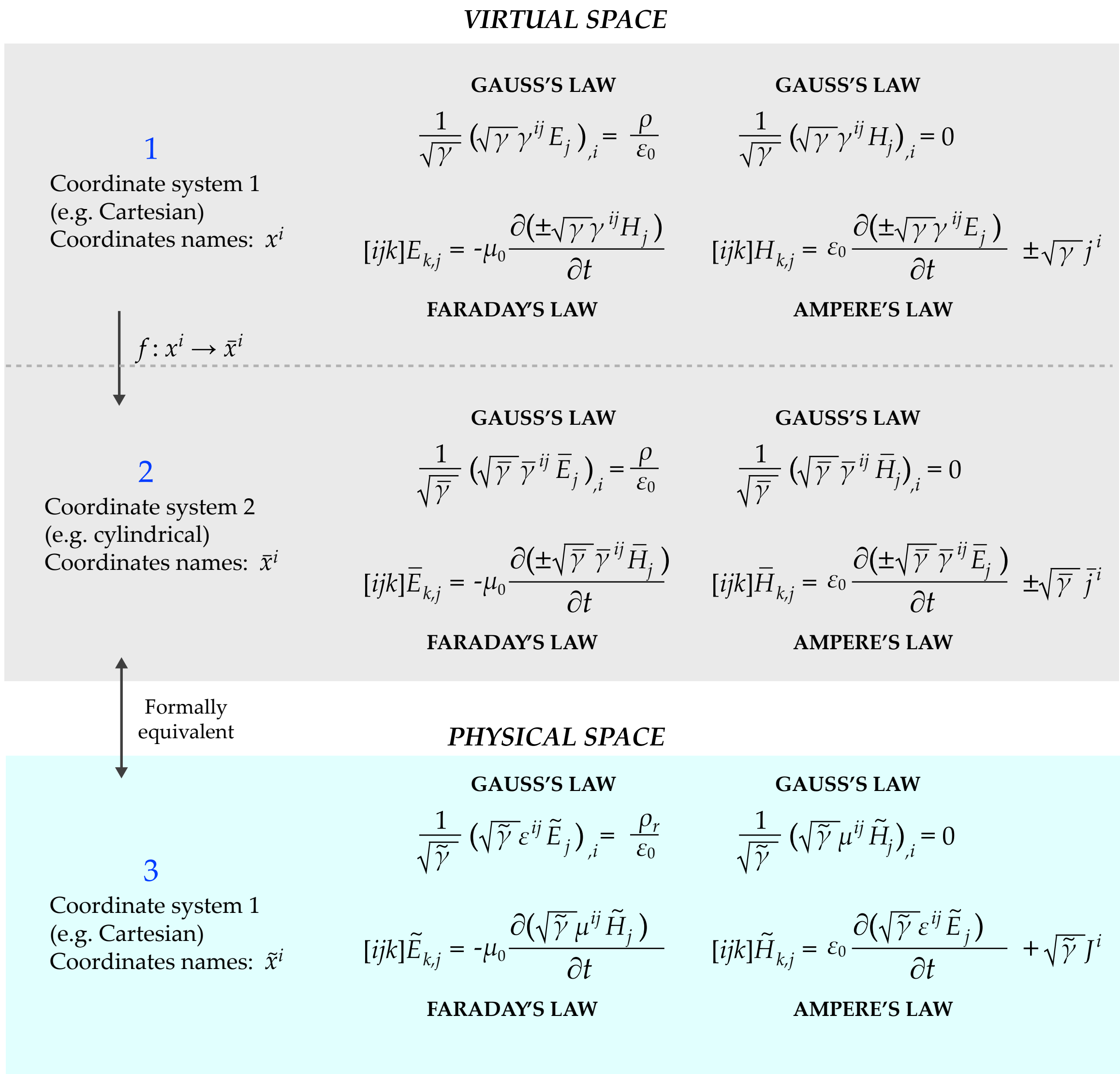

Figure 1: Transformation optics scheme

deriving the sound equation in general and the concept of acoustic metafluid. After that we will give a description STA and we will point out some of its shortcomings.

3.1 Linear acoustic equations and the wave equation for the pressure

We now proceed to review briefly the basic equations on which the standard approach to transformation acoustics is based. Given a fluid characterized by a density ρ, a pressure p, an entropy s and a velocity v with respect to a reference inertial observer, the equations characterizing its dynamics are [21]:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = M, \qquad \text{(continuity equation)}, \tag{5}$$

$$\rho D\mathbf{v} = -\nabla p - \rho \nabla \Phi + \mathbf{f}, \qquad \text{(Euler equation)}, \tag{6}$$

$$Ds = 0, \qquad \text{(entropy equation)}, \tag{7}$$

$$Dp = c^2 D\rho, \qquad \text{(sound speed definition)}, \tag{8}$$

$$p = f(\rho, s), \qquad \text{(equation of state)}, \tag{9}$$

where $M, \mathbf{f}, \Phi$ represent the mass source, the resultant of external (non-gravitational) forces and the gravitational potential respectively. We have also defined the operator D as $Dx = \partial_t x + \mathbf{v} \cdot \nabla x$. This operator is sometimes called material or convective derivative. In order to obtain the so-called sound equation we need to linearize the above system[3]. Restricting to adiabatic perturbations, and setting

$$\rho = \rho_0 + \epsilon \rho_1, \tag{10}$$

$$p = p_0 + \epsilon p_1, \tag{11}$$

we have the following background equations

$$\partial_t \rho_0 + \nabla \cdot (\rho_0 \mathbf{v}_0) = M \ , \tag{12}$$

$$\rho_0 D\mathbf{v}_0 = -\nabla p_0 - \rho_0 \nabla \Phi + \mathbf{f} \ , \tag{13}$$

$$Dp_0 = c_0^2 D\rho_0 \ . \tag{14}$$

At first order, in the absence of flow ($\mathbf{v}_0 = 0$) and when $\Phi =$ constant and $\mathbf{f} = 0$ we can obtain a simple equation for the pressure perturbation p_1. Let us detail the calculation for the static case. We have

$$M = 0 \ , \tag{15}$$

$$0 = -\nabla p_0 - \rho_0 \nabla \Phi + \mathbf{f} \ , \tag{16}$$

$$\partial_t p_0 = c_0^2 \partial_t \rho_0 = 0 \ . \tag{17}$$

[3] Without additional information, when linearizing there is a certain ambiguity as to treating $M, \mathbf{f}, \Phi$ as fixing the background or being a source in the perturbed equations. The overall magnitude of their effects could serve to decide. Here we take the view that all of those quantities affect directly the background.

Manipulating one obtains

$$\partial_t \rho_1 + \mathbf{v}_1 \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \mathbf{v}_1 = 0 \ , \tag{18}$$

$$\rho_0 \partial_t \mathbf{v}_1 = -\nabla p_1 + \frac{\rho_1}{\rho_0} \nabla p_0 - \frac{\rho_1}{\rho_0} \mathbf{f} \ , \tag{19}$$

$$\partial_t p_1 + \mathbf{v}_1 \cdot \nabla p_0 = -c_0^2 \rho_0 \nabla \cdot \mathbf{v}_1 \ , \tag{20}$$

When $\mathbf{f} = \nabla p_0$, that is, when there are no forces associated with the density (which implies that the Newtonian potential force Φ =constant), we can find

$$\rho_0 \partial_t \mathbf{v}_1 = -\nabla p_1 \ , \tag{21}$$

$$\partial_t p_1 + \mathbf{v}_1 \cdot \mathbf{f} = -c_0^2 \rho_0 \nabla \cdot \mathbf{v}_1 \ . \tag{22}$$

If in addition we have $\mathbf{f} = 0$, we obtain

$$\rho_0 \partial_t \mathbf{v}_1 = -\nabla p_1 \ , \tag{23}$$

$$\partial_t p_1 = -c_0^2 \rho_0 \nabla \cdot \mathbf{v}_1 \ . \tag{24}$$

From here one easily sees that

$$\partial_t^2 p_1 = b_0 \nabla \cdot \left(\rho_0^{-1} \nabla p_1 \right) \ . \tag{25}$$

where b_0 is the static medium bulk modulus, defined as $b_0 = c_0^2 \rho_0$. Finally, if we assume a constant density of the background such that $\nabla \rho_0 = 0$, we obtain

$$\partial_t^2 p_1 = c_0^2 \nabla^2 p_1 \ . \tag{26}$$

This is the basic equation typically used as a starting point in STA, and we will come back to the assumptions underlying its derivation in Section 3.4.

3.2 Homogenization and metafluids

As will be explained in detail in the next section, STA uses the invariance of Eq. (26) under spatial coordinate transformations to define the properties that a fluid should have to return a desired acoustic behaviour. These specific fluids are characterized by microstructures and properties outside those found in nature and for this reason they are called acoustic metafluids [22]. Acoustic metafluids can also be defined as the class of fluids that allow one domain of fluid to acoustically mimic another and have to be designed and engineered to satisfy special requirements.

In the following we will consider the class of metafluids whose functionality is based on fluids with anisotropic mass density and scalar bulk modulus [17, 23] and in particular on how to engineer them.

It has been demonstrated that metafluids with effective mass anisotropy can be realized in two dimensions by using non-isotropic lattices (i.e., other than the square and hexagonal lattices) of solid cylinders in air or in any other conventional fluid [24]. So, if we consider the propagation of sound waves in a non-isotropic periodic lattice, after a homogenization process we obtain a set of macroscopic acoustic equations [25, 26] that, using differential geometry, can be expressed as

$$p_{1,i} + \rho_{ij}\partial_t {v_1}^j = 0 \tag{27}$$

$$\partial_t p_1 + \frac{B}{\sqrt{\gamma}}\left(\sqrt{\gamma} v_1^i\right)_{,i} = 0, \tag{28}$$

which are valid when the wavelength is much larger than the lattice periodicity. Here, γ_{ij} is the three-dimensional metric, and $\gamma = \det(\gamma_{ij})$. In addition, B is the macroscopic bulk modulus and ρ_{ij} is the effective anisotropic mass density. Combining these last equations we arrive at the following macroscopic wave equation for the pressure

$$\partial_t^2 p_1 - \frac{B}{\sqrt{\gamma}}\left(\sqrt{\gamma}\rho^{ij} p_{1,j}\right)_{,i} = 0. \tag{29}$$

This equation is equal to the one derived for the propagation of sound in fluids with scalar bulk modulus and anisotropic inertia.

Currently, it is possible to realize artificial fluids in 2D with anisotropic inertia just by using non-isotropic lattices of solid rods in air [24]. Another possibility is using the homogenization properties of multilayers of two isotropic fluids [27]. This mechanism has been employed in the design of acoustic cloaks based on alternating layers of two isotropic fluids, one with low density (soft) and the other with high density (hard). However, these types of fluids are unavailable in nature and must be engineered using the concept of metafluid. These soft and hard fluids have not been realized yet since for their fabrication a solid material would be needed with low density and huge speed of sound [28, 29].

The practical realizations of metafluids with Cartesian and cylindrical anisotropy have been already proved in air background [30, 31, 32]. The main conclusion obtained from the parameters' behavior is that the sound speed cannot be made larger than the phase velocity of sound in open air.

For the realization of acoustic metafluids for underwater operation the range of tailoring possibilities broadens since the pressure wave enters into the sound scatterers and it is possible to play with the elastic parameters of the rods embedded in water. However, the fundamental limitation of metafluids for airborne sound still remains; it is not possible to tailor metafluids in which the phase velocity of sound is larger in the metafluid than in the background, in this case water.

3.3 Standard transformation acoustics: transforming the pressure wave equation

The standard approach to transformation acoustics is essentially inspired by the one of transformation optics. However, differently from the latter, standard transformation acoustics is based on a second order differential equation rather than a set of first order ones. This second order equation can be proven to be invariant under pure spatial transformations and the success of standard transformation acoustics depends on this property.

In order to understand how transformation acoustics works in detail, let us follow the same steps that we have presented in Section 2 (see Fig 2). Let us consider a homogeneous and isotropic medium (i.e. such that in Cartesian coordinates $\rho_{ij} = \rho_v \delta_{ij}$, ρ_v being a function of the spatial coordinates) embedded in a space with metric γ_{ij} (the "virtual space"). In this case, Eq. (29) becomes (Step 1)

$$\partial_t^2 p_1 = \frac{B_v}{\sqrt{\gamma}} \left(\frac{1}{\rho_v} \sqrt{\gamma} \gamma^{ij} p_{1,j} \right)_{,i} . \tag{30}$$

We perform a coordinate transformation to a metric $\bar{\gamma}_{ij}$. Then, Eq. (30) can be written as (Step 2)

$$\partial_t^2 \bar{p}_1 = \frac{B_v}{\sqrt{\bar{\gamma}}} \left(\frac{1}{\rho_v} \sqrt{\bar{\gamma}} \bar{\gamma}^{ij} \bar{p}_{1,j} \right)_{,i} . \tag{31}$$

At this point we make a comparison with an equation defined in a flat space (the "physical space") that includes a (meta)fluid with anisotropic mass density ρ_{ij} and a bulk modulus referred to as B_r. In general coordinates the sound wave equation in this case is written as

$$\partial_t^2 \tilde{p}_1 = \frac{B_r}{\sqrt{\tilde{\gamma}}} \left(\sqrt{\tilde{\gamma}} \rho^{ij} \tilde{p}_{1,j} \right)_{,i} . \tag{32}$$

Clearly, this equation and Eq. (31) provide the same solutions when the following equivalences are satisfied (again we make $\tilde{x} = \bar{x}$ and require that the coordinate system in Step 1 and Step 3 is the same)

$$B_r = \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\bar{\gamma}}} B_v, \tag{33}$$

$$\rho^{ij} = \frac{1}{\rho_v} \frac{\sqrt{\bar{\gamma}}}{\sqrt{\tilde{\gamma}}} \bar{\gamma}^{ij}, \tag{34}$$

Note that Eqs. (33)-(34) are a generalization of Eq. (14) in Ref. [16], and reduce to them when we use Cartesian coordinates in physical space ($\gamma = \tilde{\gamma} = 1$). This generalization is extremely useful when we want to implement transformations that are described in curvilinear coordinates, such as cylindrical or spherical cloaks. Eqs. (33)-(34) also establish the connection between a (spatially) curved space and a medium.

3.4 Limitations of the standard approach

As we have seen, the standard approach to transformation acoustic seems to work without problems. However, in its present form transformation acoustics is less powerful than transformation optics. This is essentially due to the structure of the wave equation and the practical difficulties of the technological realization of the designed (meta)fluid. Let us look more closely at these issues.

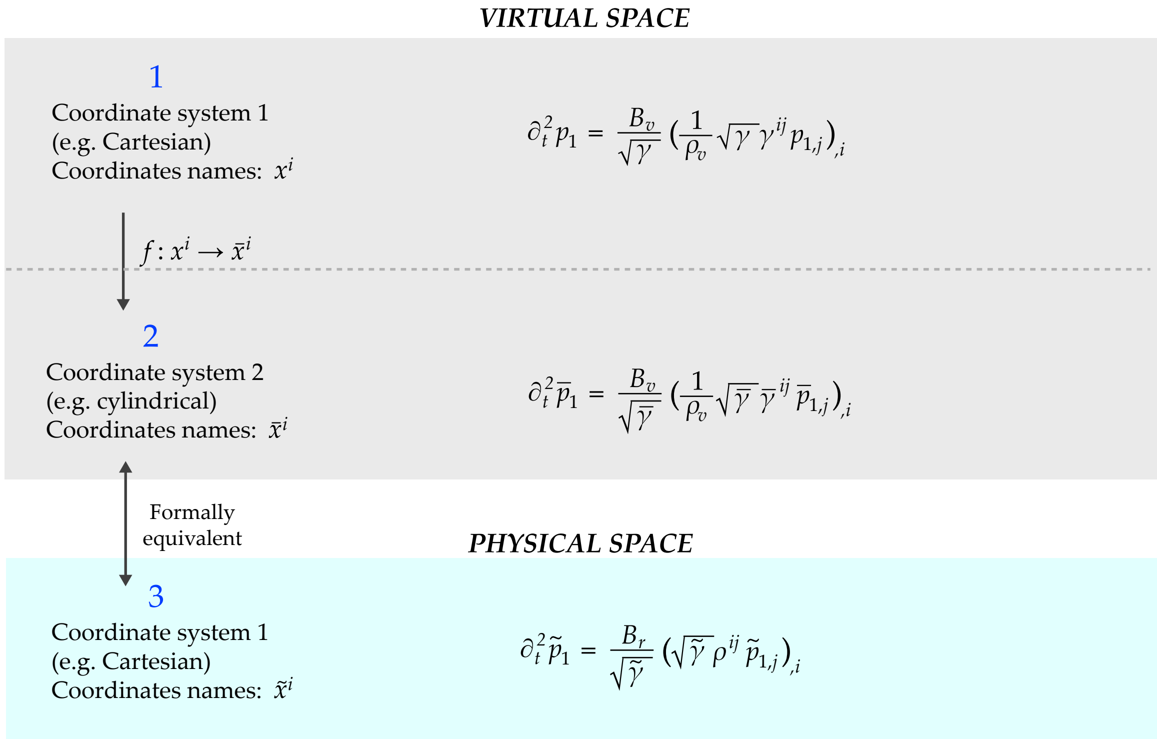

Figure 2: Standard transformation acoustics scheme

3.4.1 Restrictions of the acoustic wave equation for the pressure

If we look carefully at the derivation given in section 3.1, it is easy to realize that to arrive at the standard form of the sound wave equation, we assumed that $\nabla p_0 = 0$ and $\mathbf{v}_0 = 0$. Each of these assumptions imply a restriction to our physical model:

- In the usual derivation of a wave equation for the pressure, one assumes that the fluid is barotropic, i.e. that $p = p(\rho)$. In that case, the condition $\nabla p_0 = 0$ implies that $\nabla \rho_0 = 0$. Therefore, within the standard assumptions, he pressure wave equation needs the assumption of homogeneity of the background.

  Note that, in the derivation of the pressure wave equation, it is not strictly necessary to assume global barotropicity [33]. This condition can be relaxed and it suffices to assume a locally barotropic equation of state $p_x = p_x(\rho_x)$, or equivalently $\rho = \rho(p, x)$. One can then in principle produce arbitrary background functions $\rho_0(x)$ and $c_0(x)$. It should be stressed that $\nabla p_0 = 0$ must still hold in order to arrive at the pressure wave equation (although this no longer entails that $\nabla \rho_0 = 0$ everywhere). Also, care must be taken in the formal treatment, since the x dependences in $\rho_0(x)$ and $c_0(x)$ are not strictly independent. Moreover, such arbitrary background functions based on a spatially varying barotropic relation will be hard to reproduce with realistic fluids, especially if one assumes that the temperature distribution is approximately uniform throughout the system.

- The condition $\mathbf{v}_0 = 0$[4] implies that the formalism is limited to non-moving media (static background).

3.4.2 Implementation difficulty of the required metafluid

For the sake of simplicity, to illustrate this point we will restrict ourselves to conformal transformations, which require only isotropic media for their implementation (provided that the medium in virtual space is also isotropic, which is the usual case). Thus, the metrics in virtual and physical space will be of the form

$$\bar{\gamma}_{ij} = \Omega_v \delta_{ij} \Rightarrow \bar{\gamma} = \Omega_v^3, \tag{35}$$

$$\tilde{\gamma}_{ij} = \Omega_r \delta_{ij} \Rightarrow \tilde{\gamma} = \Omega_r^3. \tag{36}$$

In this case, we do not need to work with the homogenized wave equation. We just need to particularize the wave equation Eq. (25) for virtual and physical spaces, which in index notation read

$$\partial_t^2 \bar{p}_1 = \frac{B_v}{\sqrt{\bar{\gamma}}} \left(\frac{1}{\rho_v} \sqrt{\bar{\gamma}} \bar{\gamma}^{ij} \bar{p}_{1,j} \right)_{,i}, \tag{37}$$

$$\partial_t^2 \tilde{p}_1 = \frac{B_r}{\sqrt{\tilde{\gamma}}} \left(\frac{1}{\rho_r} \sqrt{\tilde{\gamma}} \tilde{\gamma}^{ij} \tilde{p}_{1,j} \right)_{,i}, \tag{38}$$

[4] When putting $\mathbf{v}_0 = 0$ any perturbative expansion would have to be built for v_1/c_0 small and not for v_1/v_0 small, which makes no sense.

where B_v and ρ_v are the bulk modulus and mass density in virtual space and B_r and ρ_r are the corresponding parameters in real space. These equations have the same solutions if

$$\frac{B_v}{\sqrt{\bar{\gamma}}} = \frac{B_r}{\sqrt{\tilde{\gamma}}}, \tag{39}$$

$$\frac{1}{\rho_v}\sqrt{\bar{\gamma}}\bar{\gamma}^{ij} = \frac{1}{\rho_r}\sqrt{\tilde{\gamma}}\tilde{\gamma}^{ij}, \tag{40}$$

Using Eqs. (35)-(36), we arrive at

$$B_r = B_v\frac{\Omega_r^{3/2}}{\Omega_v^{3/2}}, \tag{41}$$

$$\rho_r = \rho_v\frac{\Omega_r^{1/2}}{\Omega_v^{1/2}}, \tag{42}$$

In order to compare this approach with the one that we will propose below, it will be interesting to know the local sound speed $c = \sqrt{b/\rho}$ and acoustic impedance $z = \rho c$. Using the previous equations

$$c_r^2 = \frac{B_r}{\rho_r} = c_v^2\frac{\Omega_r}{\Omega_v}, \tag{43}$$

$$z_r = \rho_r c_r = z_v\frac{\Omega_r}{\Omega_v}. \tag{44}$$

As we can see, the mass density and speed of sound of virtual space are multiplied by the same factor, i.e., they both decrease or increase simultaneously. However, unlike natural media, the majority of artificial composites follow a different behavior, with sound speed increasing when the density decreases and vice versa [34, 20]. This holds for different background fluids such as air and water [35]. This fact complicates the implementation of the properties required to achieve acoustic devices based on this transformation acoustics approach.

4 The velocity potential

An alternative form of writing down a sound wave equation is to consider, instead of the pressure, the velocity potential that is defined as $\mathbf{v} = -\nabla\phi$. The use of this function, rather than the pressure, will be crucial for our purposes. We will now derive the evolution equation for ϕ and in the next sections we will show how we can construct on the base of this function a new approach to transformation acoustics.

We start from the continuity and Euler equations, Eqs. (5)-(6), with $M = \mathbf{f} = 0$ and Φ=constant

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{45}$$

$$\rho\left[\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] = -\nabla p. \tag{46}$$

We use the identity

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v}=\nabla\left(\frac{1}{2}\mathbf{v}^2\right)-\mathbf{v}\times(\nabla\times\mathbf{v})\,, \tag{47}$$

to transform Euler's equation to

$$\partial_t\mathbf{v}=\mathbf{v}\times(\nabla\times\mathbf{v})-\frac{1}{\rho}\nabla p-\nabla\left(\frac{1}{2}v^2\right). \tag{48}$$

If the flow is irrotational ($\nabla\times\mathbf{v}=0$), the last equation reduces to

$$\partial_t\mathbf{v}=-\frac{1}{\rho}\nabla p-\nabla\left(\frac{1}{2}v^2\right). \tag{49}$$

If the fluid moreover is barotropic ($\rho=\rho\,(p)$), we can define the enthalpy as

$$h\,(p)=\int_0^p\frac{dp'}{\rho\,(p')}. \tag{50}$$

Considering that

$$\frac{dh\,(p)}{dp}=\frac{d}{dp}\int_0^p\frac{dp'}{\rho\,(p')}=\frac{1}{\rho\,(p)}, \tag{51}$$

Eq. (50) implies that

$$\nabla h\,(p)=\frac{dh\,(p)}{dp}\nabla p=\frac{1}{\rho\,(p)}\nabla p. \tag{52}$$

As a consequence, Eq. (49) can be expressed as

$$\partial_t\mathbf{v}=-\nabla h-\nabla\left(\frac{1}{2}v^2\right). \tag{53}$$

Using the relation $\mathbf{v}=-\nabla\phi$, Euler's equation finally becomes

$$-\nabla\partial_t\phi=-\nabla h-\nabla\left(\frac{1}{2}(\nabla\phi)^2\right)\Rightarrow-\partial_t\phi+h+\frac{1}{2}(\nabla\phi)^2=0. \tag{54}$$

Now we proceed to linearize the equation of continuity and this last version of Euler's equation. As before, we express the variables involved in these equations as $\rho=\rho_0+\epsilon\rho_1$, $p=p_0+\epsilon p_1$, and $\phi=\phi_0+\epsilon\phi_1$, where the subscripts 0 and 1 indicate ambient values (in the absence of acoustic perturbations) and their fluctuations (due to a propagating acoustic wave), respectively. We introduce a similar definition for the velocity based on the linearization of the potential

$$-\nabla\phi=-\nabla\phi_0+\epsilon\,(-\nabla\phi_1)\Rightarrow\mathbf{v}=\mathbf{v_0}+\epsilon\mathbf{v_1}. \tag{55}$$

Inserting these linearized variables into the continuity equation we obtain

$$\partial_t \rho_0 + \epsilon \partial_t \rho_1 + \nabla \cdot (\rho_0 \mathbf{v_0}) + \epsilon \nabla \cdot (\rho_0 \mathbf{v_1} + \rho_1 \mathbf{v_0}) + \epsilon^2 \nabla \cdot (\rho_1 \mathbf{v_1}) = 0. \tag{56}$$

Neglecting second-order terms, we obtain two equations, one for the background and another for the acoustic perturbation

$$\partial_t \rho_0 + \nabla \cdot (\rho_0 \mathbf{v_0}) = 0, \tag{57}$$

$$\partial_t \rho_1 + \nabla \cdot (\rho_0 \mathbf{v_1} + \rho_1 \mathbf{v_0}) = 0. \tag{58}$$

In addition, we use the first order Taylor expansion of $h\,(p)$

$$h\,(p) = h\,(p_0 + \epsilon p_1) = h\,(p_0) + (\epsilon p_1)\left.\frac{\partial h\,(p)}{\partial p}\right|_{p=p_0} = h_0 + (\epsilon p_1)\left.\frac{1}{\rho\,(p)}\right|_{p=p_0} = h_0 + \epsilon \frac{p_1}{\rho_0}, \tag{59}$$

to linearize Eq. (54), which becomes

$$-\partial_t \phi_0 - \epsilon \partial_t \phi_1 + h_0 + \epsilon \frac{p_1}{\rho_0} + \frac{1}{2}(\nabla \phi_0)^2 + \epsilon \overbrace{\nabla \phi_0}^{-\mathbf{v_0}} \cdot \nabla \phi_1 + \epsilon^2 \frac{1}{2} (\nabla \phi_1)^2 = 0. \tag{60}$$

Again, neglecting second-order terms, we arrive at two equations (ambient and perturbation)

$$-\partial_t \phi_0 + h_0 + \frac{1}{2}(\nabla \phi_0)^2 = 0, \tag{61}$$

$$-\partial_t \phi_1 + \frac{p_1}{\rho_0} - \mathbf{v_0} \cdot \nabla \phi_1 = 0 \Rightarrow p_1 = \rho_0 \left(\partial_t \phi_1 + \mathbf{v_0} \cdot \nabla \phi_1\right). \tag{62}$$

The second equation above gives us a relation between ϕ_1 and p_1

$$p_1 = \rho_0 (\partial_t + v_0^i \nabla_i) \phi_1 \ . \tag{63}$$

This equation will have an important role in the next sections. From the expansions of ρ and p we obtain the following relation

$$\rho_1 = \frac{d\rho_0}{dp_0} p_1 \ = \frac{1}{c_0^2} p_1, \tag{64}$$

where c_0 is the background speed of sound, defined as $c_0^{-2} = d\rho_0 / dp_0$. Insertion of Eq. (62) into Eq.(64) leads to

$$\rho_1 = \frac{d\rho_0}{dp_0} \rho_0 \left(\partial_t \phi_1 + \mathbf{v_0} \cdot \nabla \phi_1\right). \tag{65}$$

Finally, we substitute Eq. (65) into the linearized equation of continuity Eq. (58) to obtain a wave equation for the velocity potential

$$-\frac{\partial}{\partial t}\left(\frac{d\rho_0}{dp_0}\rho_0 \left(\partial_t \phi_1 + \mathbf{v_0} \cdot \nabla \phi_1\right)\right) + \nabla \cdot \left(\rho_0 \nabla \phi_1 - \frac{d\rho_0}{dp_0} \rho_0 \left(\partial_t \phi_1 + \mathbf{v_0} \cdot \nabla \phi_1\right) \mathbf{v_0}\right) = 0. \tag{66}$$

The above equation represents the general evolution equation for the potential ϕ in the case of a barotropic fluid moving with velocity $\mathbf{v}_0$. In the case of $\mathbf{v}_0 = 0$ the above equation reduces to

$$-\frac{\partial}{\partial t}\left(\frac{\rho_0}{c_0^2}\partial_t\phi_1\right) + \nabla\cdot\rho_0\nabla\phi_1 = 0, \tag{67}$$

which, for constant c_0 and ρ_0, is simply a standard wave equation for ϕ_1.

Our target is to use these equations to devise a new approach to transformation acoustics.

5 Analogue gravity

Eq. (66) is rather involved and one is led to ask if there is a more compact and elegant way of writing it. The answer comes from a completely different research field: the one of analogue gravity (see [36] for a review).

Over the past decade, analogue models of gravity have attracted considerable attention by successfully applying differential-geometric methods known from general relativity to seemingly disconnected research areas, among them condensed matter systems, fluid- and hydro-dynamics, and acoustics. A surprising fact is that many of such physical models display an underlying Riemannian structure which can be exploited to gain new theoretical insights and to make useful experimental predictions in a laboratory environment. Lorentzian differential geometry thus also proves to be a fruitful framework beyond its traditional realm of Einstein's general relativity.

One of the principal aims of analogue gravity is to develop experiments of relativistic fields in curved spacetime backgrounds, for example related to phenomena associated to black holes such as Hawking radiation [37, 38, 39, 40]. Some of the simplest analogue gravity models deal with the acoustic propagation of sound waves in certain types of fluids [41]. Fluids which are approximately barotropic, inviscid and irrotational can be shown to provide an effective Lorentzian metric for the propagation of such perturbations [42]. Mathematically, this is possible because one can prove that if one has a system of PDEs that can be written in first-order quasi-linear symmetric hyperbolic form, then it is an exact non-perturbative result that the matrix of coefficients for the first-derivative terms can be used to construct a conformal class of metrics that encodes the causal structure of the system of PDEs [36].

In the case of acoustics one can easily prove that the wave equation Eq. (66) can be written as

$$\partial_t f^{00}\partial_t\phi_1 + \partial_t f^{0i}\nabla_i\phi_1 + \nabla_i f^{i0}\partial_t\phi_1 + \nabla_i f^{ij}\nabla_j\phi_1 = 0\ ; \tag{68}$$

with

$$f^{\mu\nu} = \frac{\rho_0}{c_0^2}\left(\begin{array}{ccc} -1 & \vdots & -v_0^i \\ \cdots & \cdot & \cdots\cdots\cdots \\ -v_0^i & \vdots & (c_0^2\delta^{ij} - v_0^i v_0^j) \end{array}\right), \tag{69}$$

where the Greek indices run from 0 to 3, while Latin indices run from 1 to 3. Then, introducing $(3+1)$–dimensional space-time coordinates, which we write as $x^\mu \equiv (t; x^i)$, Eq. (68) is easily rewritten as

$$\partial_\mu (f^{\mu\nu}\, \partial_\nu \phi_1) = 0. \tag{70}$$

Comparing this equation with the expression of the d'Alembertian of a scalar field φ in a pseudo–Riemannian manifold:

$$\Delta\varphi \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\; g^{\mu\nu}\; \partial_\nu \varphi \right), \tag{71}$$

it is clear that if we choose

$$\sqrt{-g}\; g^{\mu\nu} = f^{\mu\nu}, \tag{72}$$

and, consequently,

$$\det(f^{\mu\nu}) = (\sqrt{-g})^4\; g^{-1} = g, \tag{73}$$

then

$$g = -\frac{\rho_0^4}{c^2} \qquad \sqrt{-g} = \frac{\rho_0^2}{c}, \tag{74}$$

and the metric will be

$$g^{\mu\nu}(t,\mathbf{x}) \equiv \frac{1}{\rho_0 c} \left[\begin{array}{ccc} -1 & \vdots & -v_0^j \\ \cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ -v_0^i & \vdots & (c^2\, \delta^{ij} - v_0^i\, v_0^j) \end{array} \right], \tag{75}$$

or lowering the indices

$$g_{\mu\nu}(t,\mathbf{x}) \equiv \frac{\rho_0}{c} \left[\begin{array}{ccc} -(c^2 - v_0^2) & \vdots & -v_0^j \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -v_0^i & \vdots & \delta_{ij} \end{array} \right]. \tag{76}$$

Using this metric, Eq.(66) can be considered as the equation describing a scalar field φ propagating in a (3+1)-dimensional Lorentzian geometry [36]

$$\Delta\phi_1 = 0 \Rightarrow \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi_1 \right)_{,\mu} = 0. \tag{77}$$

The possibility to write the equation for the phase perturbation ϕ_1 as a (3+1)-dimensional equation in an abstract space will facilitate its manipulation and add further insight to our new transformation acoustics approach.

6 Analogue Transformation Acoustics

The connection between the non-relativistic sound equation and the evolution of a scalar field in curved spacetimes offers a new perspective on the problem of the transformation acoustics. In particular we can use the symmetries of Eq. (77) to obtain new and more natural design prescriptions for a given behavior of sound waves. We will call this technique *Analogue Transformation Acoustics* (ATA).

In order to define ATA we need first to generalize the analogue gravity equation to a general coordinate system. We do this in the next subsection.

6.1 Generalizing the Analogue Gravity equations

In the equations of the previous section, one typically assumes that the objects are written in Cartesian coordinates so that any covariant derivative ∇_i is actually equivalent to the standard partial derivative ∂_i. One could try to write down the equations in arbitrary coordinates. For that one has to notice that the mass density and the pressure are scalars under general changes of spatial coordinates. On the other hand, v^i is a vector. Under general changes of coordinates mixing time and space, the resulting equation would in general not have the form of a scalar wave equation in a curved background. However, restricting to general spatial coordinates and fixing time to be laboratory time, it is not difficult to show that the perturbations of the phase satisfy an equation of the form

$$\partial_t f^{00}\partial_t\phi_1 + \partial_t f^{0i}\nabla_i\phi_1 + \nabla_i f^{i0}\partial_t\phi_1 + \nabla_i f^{ij}\nabla_j\phi_1 = 0 \ , \tag{78}$$

with

$$f^{\mu\nu} = \frac{\rho_0}{c_0^2}\left(\begin{array}{ccc} -1 & \vdots & -v_0^i \\ \cdots & . & \cdots\cdots\cdots\cdots \\ -v_0^i & \vdots & (c_0^2\gamma^{ij} - v_0^i v_0^j) \end{array}\right) . \tag{79}$$

Now, let us use the property

$$\nabla_i V^i = \frac{1}{\sqrt{q}}\partial_i(\sqrt{q}V^i) \ , \tag{80}$$

with q_{ij} the relevant metric. Let us also use the fact that $\sqrt{\gamma}$ is time-independent. Then, we can write

$$\frac{1}{\sqrt{\gamma}}\left(\partial_t f_a^{00}\partial_t\phi_1 + \partial_t f_a^{0i}\partial_i\phi_1 + \partial_i f_a^{i0}\partial_t\phi_1 + \partial_i f_a^{ij}\partial_j\phi_1\right) = 0 \ , \tag{81}$$

where

$$f_a^{\mu\nu} = \frac{\rho_0}{c_0^2}\sqrt{\gamma}\left(\begin{array}{ccc} -1 & \vdots & -v_0^i \\ \cdots & . & \cdots\cdots\cdots\cdots \\ -v_0^i & \vdots & (c_0^2\gamma^{ij} - v_0^i v_0^j) \end{array}\right) . \tag{82}$$

Dividing by ρ_0^2/c_0 this equation can be written as

$$\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\phi_1 = 0 \ , \tag{83}$$

$$g_{\mu\nu} = \frac{\rho_0}{c_0} \left(\begin{array}{ccc} -c_0^2 + v_0^i v_0^j \gamma_{ij} & \vdots & -v_0^j \gamma_{ij} \\ \cdots\cdots\cdots\cdots & . & \cdots\cdots \\ -v_0^j \gamma_{ij} & \vdots & \gamma_{ij} \end{array} \right) , \tag{84}$$

$$g^{\mu\nu} = \frac{1}{\rho_0 c_0} \left(\begin{array}{ccc} -1 & \vdots & -v_0^i \\ \cdots & . & \cdots\cdots\cdots \\ -v_0^i & \vdots & c_0^2 \gamma^{ij} - v_0^i v_0^j \end{array} \right) , \tag{85}$$

$$\sqrt{-g} = \sqrt{\gamma} \frac{\rho_0^2}{c_0} . \tag{86}$$

This metric is flat whenever v_0, c_0 and ρ_0 are constants[5].

When the background velocity is zero and the background quantities stationary, we can write

$$-\frac{\rho_0}{c_0^2} \partial_t^2 \phi_1 + \frac{1}{\sqrt{\gamma}} \partial_i \rho_0 \sqrt{\gamma} \gamma^{ij} \partial_j \phi_1 = 0 . \tag{87}$$

Defining $q_{ij} = \rho_0^2 \gamma_{ij}$ we have $\sqrt{q} = \rho_0^3 \sqrt{\gamma}$, and we can re-express the previous equation as

$$-\partial_t^2 \phi_1 + c_0^2 \rho_0^2 \frac{1}{\sqrt{q}} \partial_i \sqrt{q} q^{ij} \partial_j \phi_1 = 0 , \tag{88}$$

or written compactly

$$\partial_t^2 \phi_1 = c_0^2 \rho_0^2 \nabla_q^2 \phi_1 . \tag{89}$$

6.2 The method

We are now ready to define our new approach to transformation acoustics.

The procedure we propose works in the following way (see Fig. 3). We start from a solution for the pressure in a space filled with a simple isotropic homogeneous medium characterized by ρ_v and B_v (Step 1). The pressure is connected to the velocity potential by Eq. (63). The velocity potential fulfills a wave equation that can be recast in the form of a vanishing d'Alembertian (Step 2) using the analogue gravity paradigm, with an effective metric g that depends on the medium parameters and the initial coordinate system. In this abstract space-time and using the results of the previous section, we can perform a general coordinate transformation and write the equation for ϕ_1 in any other coordinate system (Step 3) we choose. This coordinate system will be characterized by a metric $\bar{g}$. Nonetheless, from the previous section we know that only transformations

[5] There might be other situations in which the metric is flat. One just has to calculate the Riemann tensor $R_{\mu\nu\sigma\lambda}$ and impose it to be zero.

of the form $f(x^i) : x^i \to \bar{x}^{\bar{i}} = \Lambda^{\bar{i}}_k x^k$, $f_0(x^0) : x^0 \to \bar{x}^{\bar{0}} = \Lambda^{\bar{0}}_0 x^0$ that do not mix space and time [6] are allowed in order to keep the analogy between abstract space-time and laboratory.

At this point, we consider another space filled with a different medium characterized by ρ_r and B_r (Step 5). The propagation of sound waves in this material can also be modeled by an equation of the type of Eq. (77) (Step 4). The relation between the material parameters needed to guarantee that the solution of the equation in Step 3 is the same as the one in Step 4 is given by the system of equations

$$\sqrt{-\bar{g}}\bar{g}^{\mu\nu} = \sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}, \qquad \bar{g}^{\bar{\mu}\bar{\nu}} = \Lambda^{\bar{\mu}}_{\mu}\Lambda^{\bar{\nu}}_{\nu} g^{\mu\nu}. \tag{90}$$

Solving this system of equations one finds the necessary relations between the material parameters ρ_v and B_v and ρ_r and B_r.

As in the case of transformation optics, the velocity potential in Step 4 is transformed in the following way

$$\tilde{\phi}_1(x^0, x^i) = \phi_1(f_0^{-1}(x^0), f^{-1}(x^i)). \tag{91}$$

Finally, we can obtain the pressure associated to the velocity potential of Step 4 by using Eq. (63) (Step 5) and find its relation with the original pressure of Step 1.

6.3 Some Remarks

There are some important remarks that are worth making before passing to more complete applications of the procedure outlined in the previous section. In particular:

- The equation for the phase does not require any of the assumptions which the pressure equation requires. In particular, the background can now be non-static ($\mathbf{v}_0 \neq 0$) and inhomogeneous ($\nabla p_0 \neq 0$ and hence $\nabla \rho_0 \neq 0$), even for globally barotropic fluids. In this sense, working with the phase equation guarantees a more general approach to transformation acoustics.

- The ATA approach is conceptually different from the standard transformation approach. In transformation optics, for example, we use the symmetries associated with the Maxwell equations and in particular the invariance under diffeomorphisms. In ATA we construct an abstract spacetime in such a way that the evolution of the sound waves in the media maps the evolution of a scalar field in this spacetime. We then make use of the (reduced) diffeomorphism invariance in this abstract space to deduce the desired relations between the material parameters.

- One must bear in mind that the analogue gravity equations are essentially a recasting of the equation for the scalar potential ϕ, and therefore they do not add anything new to the physics contained in the standard equations. However the analogue gravity formalism facilitates the manipulation of the equations and provides

[6] The set of transformation could be extended, but here we will limit ourselves only to the ones above for simplicity sake.

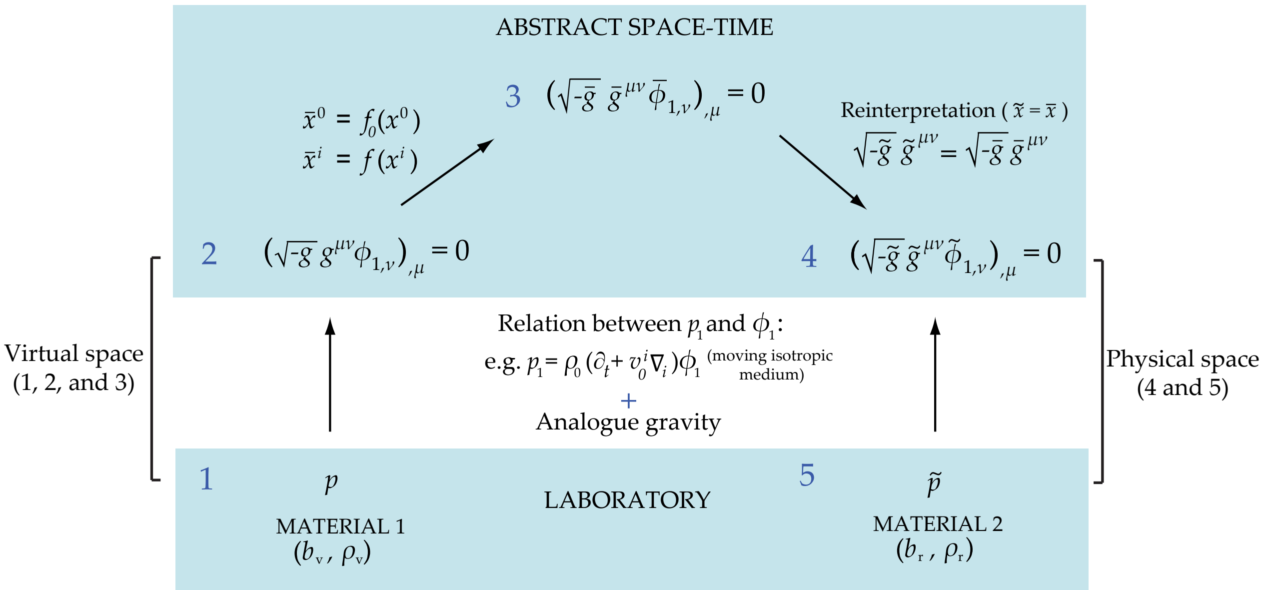

Figure 3: Proposed Analogue Transformation Acoustics (ATA) scheme. Steps 1-3 correspond to what in standard transformation acoustics and transformation optics is referred to as "virtual space". Instead, Steps 4-5 correspond to the "physical space".

us with a new point of view, allowing us to interpret the sound wave equation in terms of the kinematics of a particle in a four-dimensional spacetime. Such analogy provides a relatively easy way to treat complicated cases like the one of the moving metafluids and to transfer into acoustic metamaterials interesting phenomena typical of relativistic gravitation such as black holes, trapped surfaces etc.

7 Conformal transformations: comparison between STA and ATA

In this section, we find expressions for the material properties B_r and ρ_r that implement the transformation of the potential (parameters of the metric in Step 4) and derive the corresponding relation between the pressures in Steps 1 and 5[7]. Since the wave equation for the velocity potential does not account for anisotropy, we will restrict ourselves to conformal transformations, which preserve isotropy (in the last section we comment on how this limitation can be overcome). In addition, we assess the suitability of the proposed approach and compare it with standard transformation acoustics (based on transformations over the wave equation for the pressure), highlighting the advantages of the new method. We will consider three different cases based on pure spatial transformations: 3D conformal mappings and static background, 2D conformal mappings and static background, and general conformal mappings and moving background.

7.1 2D conformal mappings and static background

Let us start by analyzing the case of two-dimensional conformal mappings in a non-moving background. 2D conformal mappings are interesting because any (analytic) function of complex variables defines a conformal mapping. In addition, numerical methods for calculating quasi-conformal mappings have been developed mainly for two-dimensional transformations. Assuming that the original coordinate system is the Cartesian one, we have $\gamma_{ij} = \tilde{\gamma}_{ij} = \delta_{ij}$. In addition, if we leave the z variable unchanged so that

$$\bar{x} = \bar{x}(x, y), \tag{92}$$
$$\bar{y} = \bar{y}(x, y), \tag{93}$$
$$\bar{z} = z, \tag{94}$$

and use the properties $\partial x/\partial \bar{x} = \partial y/\partial \bar{y}$ and $\partial x/\partial \bar{y} = -\partial y/\partial \bar{x}$ of a conformal transformation, we obtain from the second expression in Eq. (90)

$$\bar{g}^{\mu\nu} = \frac{1}{\rho_v c_v} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{-c_v^2}{F(\bar{x},\bar{y})} & 0 & 0 \\ 0 & 0 & \frac{-c_v^2}{F(\bar{x},\bar{y})} & 0 \\ 0 & 0 & 0 & -c_v^2 \end{pmatrix} \Rightarrow \sqrt{-\bar{g}} = \frac{\rho_r^2}{c_r} F(\bar{x}, \bar{y}), \tag{95}$$

[7] To keep the same notation for STA and ATA and allow for a clearer comparison, from now on we represent the bulk modulus with a capital "B", whether it refers to a homogenized quantity or not.

with

$$F(\bar{x},\bar{y}) = \left(\frac{\partial x}{\partial \bar{x}}\right)^2 + \left(\frac{\partial y}{\partial \bar{x}}\right)^2, \tag{96}$$

On the other hand, using the expression of $\tilde{\gamma}_{ij}$

$$\tilde{g}^{\mu\nu} = \frac{1}{\rho_r c_r}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -c_r^2 & 0 & 0 \\ 0 & 0 & -c_r^2 & 0 \\ 0 & 0 & 0 & -c_r^2 \end{pmatrix} \Rightarrow \sqrt{-\tilde{g}} = \frac{\rho_r^2}{c_r}. \tag{97}$$

At first sight, it seems that the spatial part of Eq.(95) does not fulfill the isotropic requirement, since we do not have $g^{11} = g^{22} = g^{33}$. However, in our 2D problem, we assume that the velocity potential is invariant in the z-direction, so that $\phi_1 = \phi_1(x,y)$. Taking this fact into account, from Eq. (83) we observe that the value of g^{33} is irrelevant, as it is multiplied by $\partial_3\phi_1 = \partial_z\phi_1 = 0$. Thus, we have an effective 2D isotropic spatial metric, since $g^{11} = g^{22}$. In this particular case, substituting Eqs. (95)-(97) into the first expression in Eq. (90) we obtain the following two equations

$$\frac{\rho_v^2}{c_v}F(\bar{x},\bar{y})\frac{c_v}{\rho_v F(\bar{x},\bar{y})} = \frac{\rho_r^2}{c_r}\frac{c_r}{\rho_r} \Rightarrow \rho_r = \rho_v, \tag{98}$$

$$\frac{\rho_v^2}{c_v}F(\bar{x},\bar{y})\frac{1}{\rho_v c_v} = \frac{\rho_r^2}{c_r}\frac{1}{\rho_r c_r} \Rightarrow B_r = \frac{B_v}{F(\bar{x},\bar{y})}. \tag{99}$$

This means that we only need to construct a spatially varying bulk modulus to implement a two-dimensional conformal transformation.[8] It is interesting to note that for two-dimensional conformal mappings, the parameters required by STA are the same as the ones given by Eqs. (98)-(99). Thus in this case STA and ATA give the same results.

7.2 3D conformal mappings and static background

We will now consider three-dimensional transformations of a non-moving medium ($\mathbf{v}_0 = 0$) which preserve isotropy. In this case, as in Eqs. (35)-(36) we have (using also Eq. (86))

$$\bar{\gamma}_{ij} = \Omega_v \delta_{ij} \Rightarrow \sqrt{-\bar{g}} = \Omega_v^{3/2}\frac{\rho_v^2}{c_v}, \tag{100}$$

$$\tilde{\gamma}_{ij} = \Omega_r \delta_{ij} \Rightarrow \sqrt{-\tilde{g}} = \Omega_r^{3/2}\frac{\rho_r^2}{c_r}, \tag{101}$$

and the initial and final effective metrics will have the following form

[8] Note that, in general, we have an additional degree of freedom and the expression for ρ_r is $\rho_r = \alpha\rho_v$, α being a constant, since this condition also satisfies the wave equation. Nonetheless, in practice we will employ a (continuous) transformation different from the identity only in a finite region. Since we usually want to keep the background properties outside this region (i.e. $\rho_r = \rho_v \to \alpha = 1$) and α is a global constant, we will have $\alpha = 1$ everywhere in most cases of interest.

$$(\bar{g}_{\mu\nu}) = \frac{\rho_v}{c_v}\begin{pmatrix} -c_v^2 & 0 & 0 & 0 \\ 0 & \Omega_v & 0 & 0 \\ 0 & 0 & \Omega_v & 0 \\ 0 & 0 & 0 & \Omega_v \end{pmatrix} \Rightarrow (\bar{g}^{\mu\nu}) = \frac{1}{\rho_v c_v}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{c_v^2}{\Omega_v} & 0 & 0 \\ 0 & 0 & \frac{c_v^2}{\Omega_v} & 0 \\ 0 & 0 & 0 & \frac{c_v^2}{\Omega_v} \end{pmatrix}, \tag{102}$$

$$(\tilde{g}_{\mu\nu}) = \frac{\rho_r}{c_r}\begin{pmatrix} -c_r^2 & 0 & 0 & 0 \\ 0 & \Omega_r & 0 & 0 \\ 0 & 0 & \Omega_r & 0 \\ 0 & 0 & 0 & \Omega_r \end{pmatrix} \Rightarrow (\tilde{g}^{\mu\nu}) = \frac{1}{\rho_r c_r}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{c_r^2}{\Omega_r} & 0 & 0 \\ 0 & 0 & \frac{c_r^2}{\Omega_r} & 0 \\ 0 & 0 & 0 & \frac{c_r^2}{\Omega_r} \end{pmatrix}. \tag{103}$$

Using the first expression in Eq. (90), we obtain that the equations in Steps 3 and 4 have the same solutions if

$$\frac{\sqrt{-\bar{g}}}{\rho_v c_v} = \frac{\sqrt{-\tilde{g}}}{\rho_v c_v}, \tag{104}$$

$$\frac{c_v^2}{\Omega_v} = \frac{c_r^2}{\Omega_r}. \tag{105}$$

Notice that these relations are a function of Ω_v and Ω_r and are valid for general spatial conformal transformations. To find Ω_v we must use the second relation in Eq. (90) for a particular transformation. On the other hand, Ω_r will be determined by the coordinate system employed in physical space. Using the definition of bulk modulus introduced in section 3.1 and with the help of Eqs. (100)-(101), we arrive at

$$B_r = \rho_r c_r^2 = B_v \frac{\Omega_r^{1/2}}{\Omega_v^{1/2}}, \tag{106}$$

$$\rho_r = \rho_v \frac{\Omega_v^{1/2}}{\Omega_r^{1/2}}. \tag{107}$$

In terms of the speed of sound and the acoustic impedance, in this case we have

$$c_r^2 = c_v^2 \frac{\Omega_r}{\Omega_v}, \tag{108}$$

$$z_r = z_v. \tag{109}$$

Comparing with eqs. (43) and (44), it is clear that the material parameters required are different compared to STA. In particular, the transformation of the speed of sound is the same as the one obtained within STA, whereas for the impedances it is different. It is interesting to note that, if one compares the transformation laws obtained above with the ones of transformational optics, only the transformations given by ATA are structurally similar. Another important point is that the transformations obtained using ATA imply that an increase of the speed of sound corresponds to a decrease in the density of the metafluid. As we mentioned in section 3.4.2, such properties are much easier to implement from a technological point of view than the ones obtained by STA.

Let us see how the pressure is transformed in each approach. For the static case that we are considering ($\mathbf{v}_0 = 0$), the pressure and the potential are related by

$$p_1 = \rho_0 \frac{\partial \phi_1}{\partial t}. \tag{110}$$

In STA, we directly transform the equation for the pressure. Thus, since the pressure transforms as a scalar, the pressure perturbation in real space $\tilde{p}_1$ (Step 5) is related to the pressure perturbation in virtual space p_1 (Step 1) as

$$\tilde{p}_1(t, x^i) = p_1(t, f^{-1}(x^i)), \tag{111}$$

where f is the spatial coordinate transformation performed to change from Step 2 to Step 3. However, in ATA, we apply the transformation to the wave equation for the potential, so we have

$$\tilde{\phi}_1(t, x^i) = \phi_1(t, f^{-1}(x^i)), \tag{112}$$

where ϕ_1 and $\tilde{\phi}_1$ are the potentials in Steps 2 and 4, respectively. Now, using Eq. (110), we know that

$$p_1(t, x^i) = \rho_v(x^i) \frac{\partial \phi_1(t, x^i)}{\partial t}, \tag{113}$$

and therefore

$$\tilde{p}_1(t, x^i) = \rho_r(x^i) \frac{\partial \tilde{\phi}_1(t, x^i)}{\partial t} = \rho_v(f^{-1}(x^i)) \frac{\Omega_v^{1/2}}{\Omega_r^{1/2}} \frac{\partial \phi_1(t, f^{-1}(x^i))}{\partial t} = p_1(t, f^{-1}(x^i)) \frac{\Omega_v^{1/2}}{\Omega_r^{1/2}}. \tag{114}$$

Comparing with the corresponding transformation in STA, Eq. (111), we can see that the pressure in ATA is further corrected by a factor $\frac{\Omega_v^{1/2}}{\Omega_r^{1/2}}$.

7.2.1 Example: uniform scaling and plane waves

As an example, let us consider the most simple conformal transformation, a uniform spatial scaling

$$\bar{x}^0 = x^0, \tag{115}$$

$$\bar{x}^i = A x^i, \tag{116}$$

where A is an arbitrary constant. In addition, we will use Cartesian coordinates in Steps 2 and 4, so that

$$\gamma^{ij} = \delta^{ij} \Rightarrow \sqrt{\gamma} = 1, \tag{117}$$

$$\tilde{\gamma}^{ij} = \delta^{ij} \Rightarrow \sqrt{\tilde{\gamma}} = 1 \Rightarrow \Omega_r = 1. \tag{118}$$

Thus, we can obtain $\bar{\gamma}^{ij}$ as

$$\bar{\gamma}^{\bar{i}\bar{j}} = \Lambda^{\bar{i}}_i \Lambda^{\bar{i}}_j \delta^{ij} = A^2 \delta^{ij} \Rightarrow \Omega_v = 1/A^2. \tag{119}$$

As a consequence, using Eqs. (106)-(107), we get the following material properties

$$B_r = AB_v, \tag{120}$$

$$\rho_r = \rho_v / A. \tag{121}$$

The corresponding sound speed is

$$c_r = Ac_v. \tag{122}$$

To understand how a plane wave is "distorted" under this simple transformation we proceed as following. In Step 1 (isotropic homogeneous medium), the wave equation satisfied by the pressure is

$$\ddot{p}_1 = \frac{B_v}{\rho_v} \nabla^2 p_1. \tag{123}$$

This equation admits plane-wave solutions of the form

$$p_1 = p_{1,0} e^{j(\mathbf{k_v} \cdot \mathbf{r} - \omega t)}, \tag{124}$$

with

$$\frac{\omega^2}{|k_v|^2} = \frac{B_v}{\rho_v}. \tag{125}$$

According to equation Eq. (114), the pressure in Step 5 will be

$$\tilde{p}_1(t, x^i) = p_1(t, f^{-1}(x^i)) \frac{\Omega_v^{1/2}}{\Omega_r^{1/2}} = p_1(t, \frac{1}{A} x^i) \frac{1}{A} = \frac{1}{A} p_{1,0} e^{j(\frac{1}{A} \mathbf{k_v} \cdot \cdot \mathbf{r} - \omega t)}. \tag{126}$$

We observe two different effects here. The first one is a scaling of the momentum k_v by a factor $1/A$. We should expect this, since we have expanded space by a factor of A. Moreover, this fact is consistent with the relation between the new parameters ρ_r, B_r and the old parameters ρ_v, B_v, which, using Eq. (125) for the new parameters, leads to

$$\frac{\omega^2}{|k_r|^2} = \frac{B_r}{\rho_r} \Rightarrow |k_r| = \omega \sqrt{\frac{\rho_r}{B_r}} = \frac{1}{A} \omega \sqrt{\frac{\rho_v}{B_v}} = \frac{|k_v|}{A}. \tag{127}$$

Second, the amplitude of the wave has also been scaled by a factor of $1/A$. This effect can be justified through energy considerations. It can be shown that the energy density of a plane pressure wave is given by the expression [43]

$$E = \frac{p^2}{\rho c^2}. \tag{128}$$

Thus, the energy density in Steps 1 and 5 is

$$E = \frac{p_1^2}{\rho_v {c_v}^2}, \tag{129}$$

$$\tilde{E} = \frac{\tilde{p}_1^2}{\rho_r {c_r}^2} = \frac{\frac{1}{A^2} p_1^2(t, A^{-1}x^i)}{\frac{\rho_v}{A} A^2 c_v^2} = \frac{1}{A^3} \frac{p_1^2(t, A^{-1}x^i)}{\rho_v {c_v}^2} = \frac{1}{A^3} E(t, A^{-1}x^i), \tag{130}$$

which means that the energy density has been scaled by a factor of $1/A^3$ when we use the new material parameters. This was expected, since we are expanding the wave in the three spatial directions by a factor of A.

7.3 General conformal mapping and moving background

Let us now consider the possibility of a non-zero background velocity in virtual space and try to find out how this velocity should be changed to implement a certain coordinate transformation. To this end, we need to express Eq. (66) in arbitrary spatial coordinates

$$-\partial_t \left(\partial_t \phi_1 + v_0^m \phi_{1,m}\right) + \frac{c_v^2}{\rho_v \sqrt{\gamma}} \left[\sqrt{\gamma} \left(\rho_v \gamma^{ij} \phi_{1,j} - {c_v^{-2}} \rho_v \left(\partial_t \phi_1 + v_0^m \phi_{1,m}\right) v_0^i\right)\right]_{,i} = 0, \tag{131}$$

where we have assumed that $c_v^{-2} = \frac{d\rho_0}{dp_0}$ is time-independent. Eq. (131) represents virtual space as in stage 2 of Fig. 3. Now we express Eq. (131) in another set of spatial coordinates (stage 3)

$$-\partial_t \left(\partial_t \bar{\phi}_1 + \bar{v}_0^m \phi_{1,m}\right) + \frac{c_v^2}{\rho_v \sqrt{\bar{\gamma}}} \left[\sqrt{\bar{\gamma}} \left(\rho_v \bar{\gamma}^{ij} \bar{\phi}_{1,j} - {c_v^{-2}} \rho_v \left(\partial_t \bar{\phi}_1 + \bar{v}_0^m \bar{\phi}_{1,m}\right) \bar{v}_0^i\right)\right]_{,i} = 0, \tag{132}$$

with

$$\bar{v}_0^{\bar{i}} = \frac{\partial \bar{x}^{\bar{i}}}{\partial x^i} v_0^i \tag{133}$$

Finally, we write the wave equation in physical space (Step 4)

$$-\partial_t \left(\partial_t \tilde{\phi}_1 + \tilde{v}_0^m \phi_{1,m}\right) + \frac{c_r^2}{\rho_r \sqrt{\tilde{\gamma}}} \left[\sqrt{\tilde{\gamma}} \left(\rho_r \tilde{\gamma}^{ij} \tilde{\phi}_{1,j} - {c_r^{-2}} \rho_r \left(\partial_t \tilde{\phi}_1 + \tilde{v}_0^m \tilde{\phi}_{1,m}\right) \tilde{v}_0^i\right)\right]_{,i} = 0, \tag{134}$$

As a result, and assuming again that $\bar{\gamma}^{ij}$ and $\tilde{\gamma}^{ij}$ are given by Eqs. (35)-(36), we find that the two last equations have the same solutions if the relations between the properties of virtual and physical space given by Eqs. (106)-(107) are fulfilled (or Eqs. (98)-(99) in the case of a two-dimensional transformation), and the background velocity in real space satisfies

$$\tilde{v}_0^i = \bar{v}_0^i, \qquad \bar{v}_0^{\bar{i}} = \frac{\partial \bar{x}^{\bar{i}}}{\partial x^i} v_0^i, \tag{135}$$

Therefore, differently from STA, which assumes a zero background velocity, ATA allows us to implement transformations of moving media and provides us with the background velocity field required to achieve a certain transformation. This is one of the main advantages of ATA.

The proof above is much easier if we work with the four-dimensional metrics of abstract space-time. In fact, particularizing Eq. (85) for virtual (Step 3) and physical (Step 4) spaces, we have

$$\bar{g}^{\mu\nu} = \frac{1}{\rho_v c_v} \begin{pmatrix} -1 & \vdots & -\bar{v}_0^i \\ \cdots & . & \cdots\cdots\cdots\cdots \\ -\bar{v}_0^i & \vdots & c_v^2 \Omega_v^{-1} \delta^{ij} - \bar{v}_0^i \bar{v}_0^j \end{pmatrix}, \qquad \tilde{g}^{\mu\nu} = \frac{1}{\rho_r c_r} \begin{pmatrix} -1 & \vdots & -\tilde{v}_0^i \\ \cdots & . & \cdots\cdots\cdots\cdots \\ -\tilde{v}_0^i & \vdots & c_r^2 \Omega_r^{-1} \delta^{ij} - \tilde{v}_0^i \tilde{v}_0^j \end{pmatrix} \tag{136}$$

Clearly, the only additional requirement to satisfy Eq. (90) (in addition to Eqs. (106)-(107)) is that $\tilde{v}_0^i = \bar{v}_0^i$, and we know from Eq. (90) that for pure spatial transformations

$$\bar{v}_0^{\bar{i}} = \bar{g}^{\bar{0}\bar{i}} = \Lambda_0^{\bar{0}} \Lambda_i^{\bar{i}} g^{0i} = \Lambda_i^{\bar{i}} v_0^i = \frac{\partial \bar{x}^{\bar{i}}}{\partial x^i} v_0^i, \tag{137}$$

which coincides with Eq. (135).

8 Numerical examples

In this section we numerically demonstrate the previous formulation of transformation acoustics for each of the studied cases.

8.1 Two-dimensional case

To verify the results of the previous section, we will design a so-called carpet cloak, a concept that originally appeared in the framework of transformation optics. Essentially, a carpet cloak makes a curved wall look flat for an external observer. Thus, we can create a bump in the wall and hide an object behind it without producing additional scattering. The advantage of the carpet cloak is that it can be obtained from a quasi-conformal transformation that introduces very small anisotropy, which can be neglected. Up to date, acoustic carpet cloaks based on quasi-conformal mappings have not been verified. On the other hand, an anisotropic version of this acoustic device was recently demonstrated both theoretically and experimentally [44]. Here, we will design and numerically verify an isotropic version of the acoustic carpet cloak. To obtain the mapping,

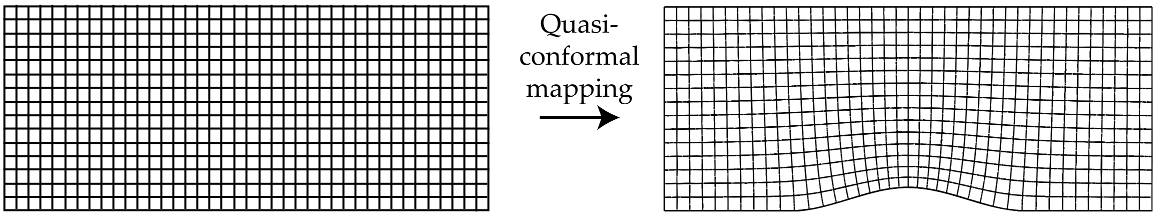

Figure 4: Carpet cloak quasi-conformal mapping

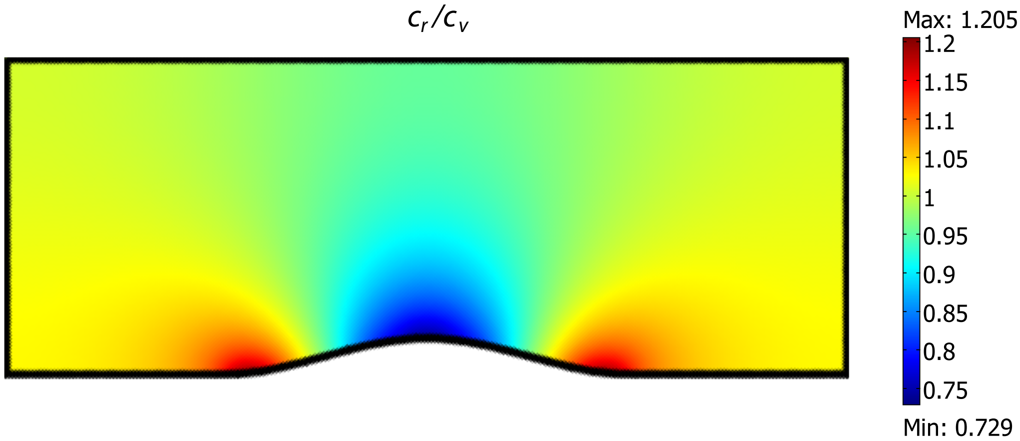

Figure 5: Relative speed of sound of the designed acoustic carpet cloak

we will use the method described in Ref. [45], from which we will obtain the function F($\bar{x}$,$\bar{y}$). The idea is to transform a rectangular region into a deformed rectangular region with a bump, as shown in Fig. 4

Using the above-mentioned algorithm and Eq. (99), we calculated the bulk modulus and speed of sound distribution associated to this transformation, which is depicted in Fig. 5.

To test the device, we simulated its performance with the commercial software COMSOL Multiphysics, both for the velocity potential wave equation as well as for the pressure wave equation. The results for the first case are shown in Fig. 6, where we compare the intensity of a Gaussian beam impinging from the left onto a flat wall, a wall with a bump, and a wall with a bump surrounded by a carpet cloak.

The velocity potential and the pressure of the simulated carpet cloak are depicted in Fig. 7.

We also solved the pressure wave equation for a carpet cloak with the same bulk modulus distribution, obtaining almost identical results, as expected based on the con-

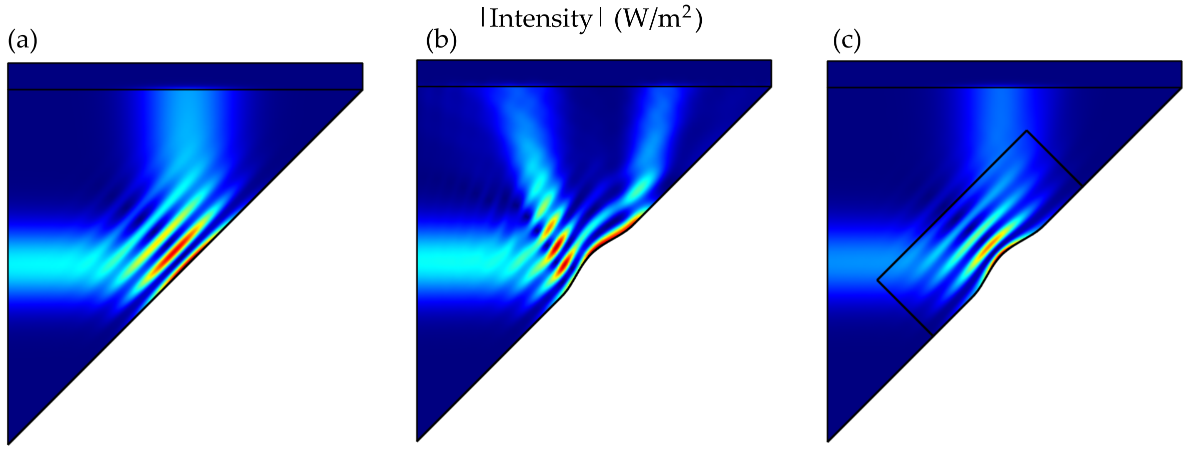

Figure 6: Simulated intensity of a Gaussian beam impinging from the left onto different objects. (a) Flat wall. (b) Wall with a bump. (c) Wall with a bump surrounded by a carpet cloak.

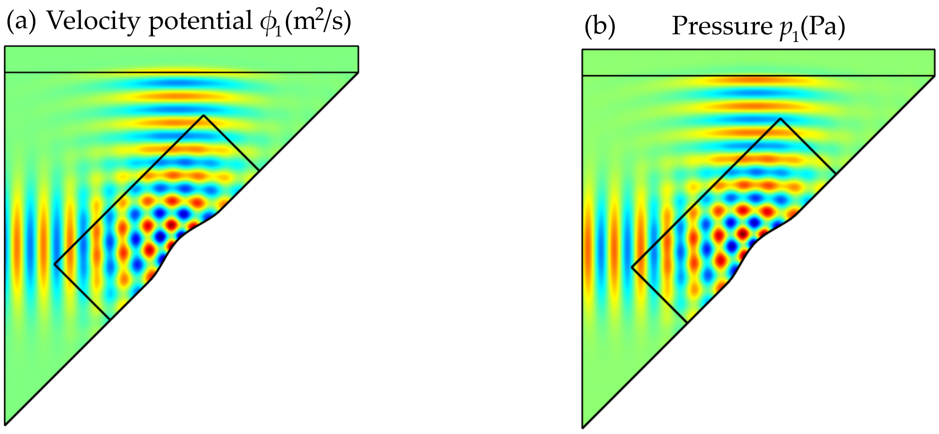

Figure 7: Velocity potential and pressure in the simulated carpet cloak.

clusions of Sec. 7.1.

8.2 Three-dimensional case

A very interesting example of an isotropic three-dimensional (3D) device is that of Maxwell's fish-eye. This element implements the mapping of the 3D surface of the four-dimensional hypersphere to the volume of a 3D sphere in flat 3D space [3, 46]. The metric of such a geometry in Cartesian coordinates is given by [3]

$$(\bar{g}_{\mu\nu}) = \frac{\rho_v}{c_v} \begin{pmatrix} -c_v^2 & 0 & 0 & 0 \\ 0 & \Omega_v & 0 & 0 \\ 0 & 0 & \Omega_v & 0 \\ 0 & 0 & 0 & \Omega_v \end{pmatrix}, \tag{138}$$

with

$$\Omega_v = \left(\frac{2}{1 + \frac{r^2}{a^2}} \right)^2, \qquad r = \sqrt{x^2 + y^2 + z^2}, \tag{139}$$

where a is the radius of the 4-sphere. Therefore, the device has an infinite extension in principle. However, it can be shown that its functionality is unaltered if we place a spherical mirror at $r = a$ [3, 46]. One of the main properties of Maxwell's fish-eye is that it focuses the rays emanating from a point source located at any arbitrary point $\mathbf{x}_0$ within the sphere to the point $-\mathbf{x}_0$ [3, 46]. Of course, this is not valid for points outside the sphere when its surface is a mirror. According to Eqs. (106)-(107), we can implement the metric given by Eq. (138) by using a medium with the following properties ($\Omega_r = 1$)

$$B_r = B_v \left(\frac{2}{1 + \frac{r^2}{a^2}} \right)^{-1}, \tag{140}$$

$$\rho_r = \rho_v \frac{2}{1 + \frac{r^2}{a^2}}. \tag{141}$$

We have verified the functionality of the acoustic Maxwell's fish-eye through three-dimensional numerical calculations as shown in Fig. 8, where we can observe its focusing properties. Being a three dimensional device, the properties required to implement Maxwell's fish-eye derived from the ATA approach are easier to achieve than those derived from the STA approach, as has been discussed above.

8.3 Transforming the background velocity

To verify the potential of working with moving media provided by our new approach, we will use the previous example of the carpet cloak, but considering a non-zero background velocity parallel to the wall. For instance, let us take $\mathbf{v}_{0v} = 50\hat{x} + 50\hat{y}$ m/s. If we use the values of ρ_r and B_r calculated for the carpet cloak in the previous section and keep

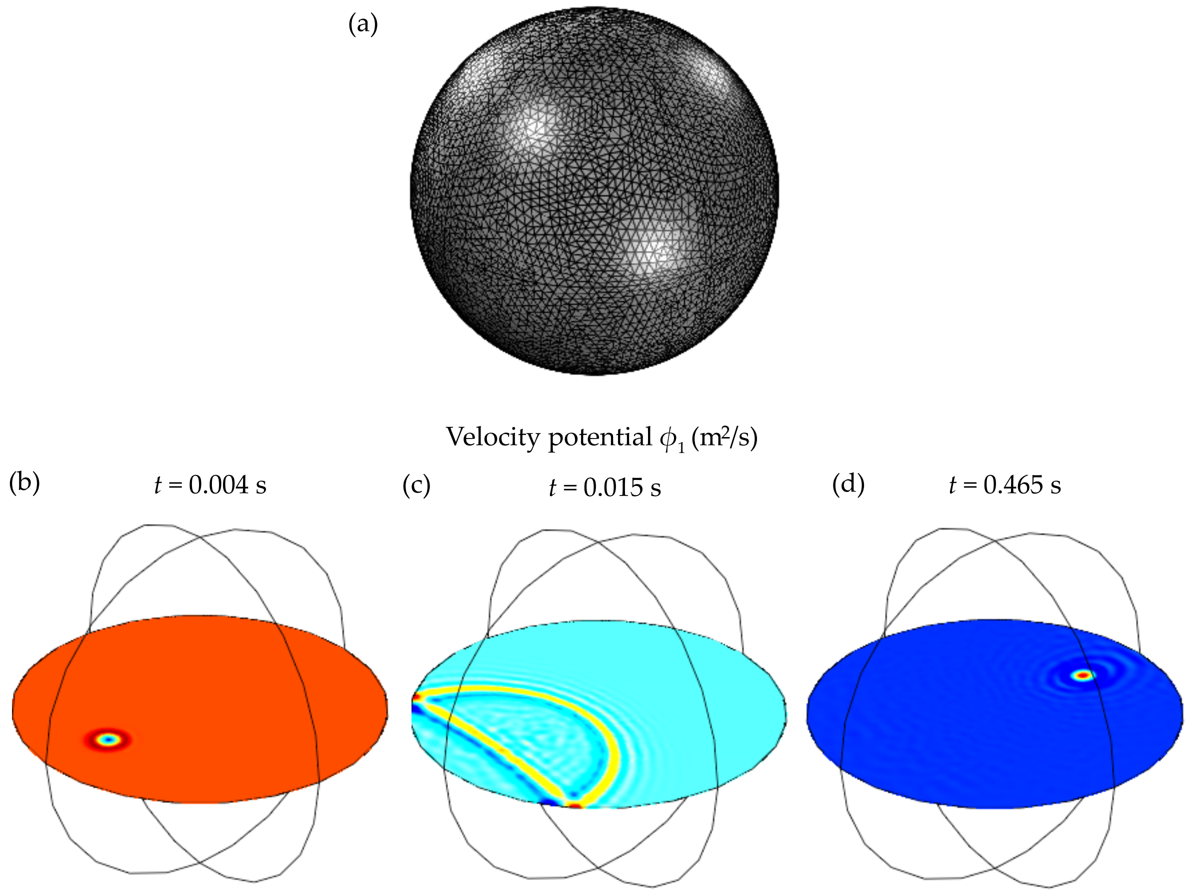

Figure 8: Three-dimensional simulation of Maxwell's fish-eye under point source excitation (a) Geometry mesh. In the simulation, the structure is surrounded by a hard wall boundary condition that acts as a mirror. A point source emits a Gaussian pulse (in the time domain) at a certain point within the spherical structure. We keep track of this wave at different instants to verify the focusing effect. In the figure, we depict the velocity potential over a spherical cross-section passing through its center at (b) t = 0.004 s, when the pulse has just been launched, (c) t = 0.015 s, and (d) t = 0.0465 s, when the pulse gets focused again. Note that, although we only show a 2D plane here, the focusing effect occurs in 3D.

this value of $\mathbf{v}_{0v}$, the wave is distorted into an undesired wave (see Fig. 9). However, if we force a background velocity given by Eq.(135) such that

$$\begin{pmatrix} \tilde{v}_0^x \\ \tilde{v}_0^y \end{pmatrix} = \begin{pmatrix} \frac{\partial \bar{x}}{\partial x} & \frac{\partial \bar{x}}{\partial y} \\ \frac{\partial \bar{y}}{\partial x} & \frac{\partial \bar{y}}{\partial y} \end{pmatrix} \begin{pmatrix} v_0^x \\ v_0^y \end{pmatrix}, \tag{142}$$

the wave is distorted in the desired way (remains unchanged outside the cloak).

9 Conclusions and Perspectives

In this report we have presented a new approach to the problem of transformational acoustics based on the analogue gravity idea. In this method one writes the acoustic equations in terms of the scalar potential ϕ and notices that the new equations allow for a more powerful approach to transformation acoustics. Successively one notices that these equations are equivalent to the equation of motion of a non-interacting scalar field in a curved spacetime. Using this analogy, one can exploit the symmetries of this fictitious spacetimes to derive the dependences of the parameters able to lead to the desired acoustic behavior of a metamaterial.

The new technique was applied to the case of transformations that preserve the isotropy of the fluid i.e. conformal transformations. In the 2D case Analogue Transformation Acoustics returns the same results of Standard Transformational Acoustics. However, in the 3D case, there are differences in these transformation laws and the results obtained with our new method appear to be more natural in terms of the technological achievement of the physical parameters of the metafluids. We also proved that it is easy to introduce, for example, a motion of the background fluid, confirming that the use of Analogue Transformation Acoustics offers clear advantages in the study of more complex settings.

Our theoretical calculations were then tested via numerical simulations. In particular we have been able to design an acoustic carpet cloak, a Maxwell Fish Eye Lens, and a carpet cloak able to compensate for the motion of the background fluid.

All these results are particularly encouraging. However, there is still much to be done before we are able to exploit fully the power of Analogue Transformation Acoustics. The most important of these possible developments are described in the sections below.

9.1 Homogenization of the phase wave equation: general spatial transformations

A first perspective for future work is to obtain an acoustic anisotropic equation starting from an inhomogeneous but isotropic fluid, for example by imposing certain periodicity conditions on the fluid.

The basic argument is the following. The wave equation for the phase in Cartesian coordinates (for a static background) reads

$$\frac{\rho_0}{c_0^2} \partial_t^2 \phi_1 = \partial_i \rho_0 \delta^{ij} \partial_j \phi_1 \ . \tag{143}$$

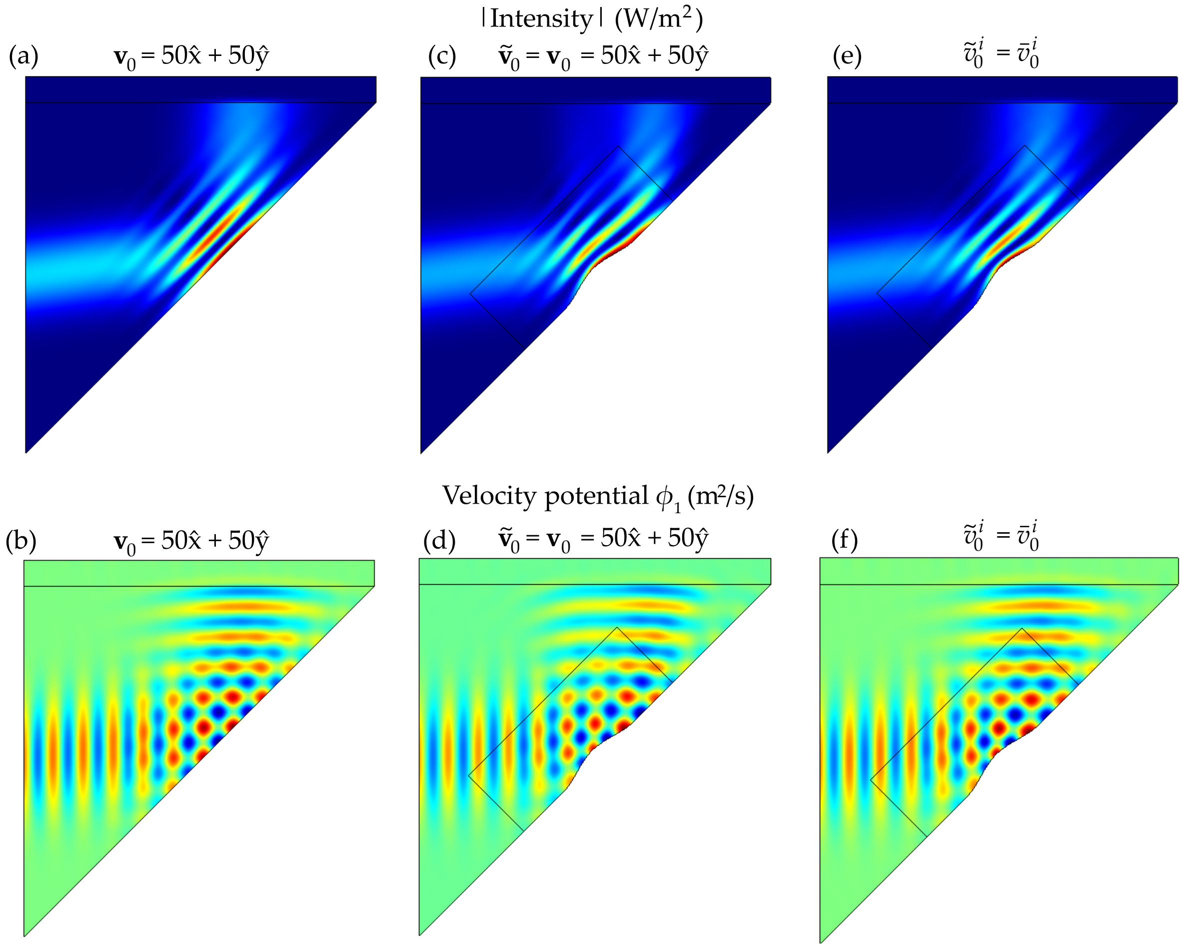

Figure 9: (a-b) Flat wall with a constant background velocity parallel to the wall (c-d) Carpet cloak with the same background velocity as in (a-b) (e-f) Carpet cloak with a properly transformed background velocity.

Imagine now that the background speed of sound c_0 and density ρ_0 are periodic functions in a Bravais lattice: $\rho_0(\mathbf{x}+\mathbf{X}) = \rho_0(\mathbf{x})$, and likewise for $c_0(x)$. Let us denote these periodic ρ_0 and c_0 by ρ_r and c_r instead.

When one is concerned with the behaviour of acoustic waves of wavelengths sufficiently large with respect to the periodic properties of the fluid, one can obtain a homogenized effective equation. It can be demonstrated mathematically [47] that, under the adequate conditions, this homogeneous macroscopic equation has the following structure:

$$\langle \frac{\rho_r}{c_r^2} \rangle \partial_t^2 \phi_1 = \partial_i \tilde{f}_r^{ij} \partial_j \phi_1 \ , \tag{144}$$

where the brackets denote average values and

$$\tilde{f}^{ij} := \langle \rho_r \rangle \delta^{ij} - \tilde{a}^{ij} \ , \tag{145}$$

with $\tilde{a}^{ij}$ a measure of the variation of the density over an average lattice period.

By adding the appropriate parameters, Eq. (144) can always be manipulated into the form of a wave equation for the phase perturbation such as eq. (87), which we used in the ATA approach. What remains to be determined, based on a careful analysis of the mathematical construction required to obtain (144) (and to derive an appropriate wave function from it), is whether and how these parameters can be matched by realistic fluid properties, and how general this procedure can be made (i.e., whether, given a particular cloaking geometry, it is always possible to construct an underlying periodic medium which would reproduce it).

Additionally, we expect that it should be relatively straightforward to generalize the procedure sketched above to incorporate a non-zero background velocity, as well as to generalize it to the case in which ρ_r and c_r are periodic locally but vary slowly throughout the lattice structure (or in which they vary slowly with time). In this way one could obtain anisotropic and inhomogeneous acoustic equations, potentially of great interest within the acoustic transformation framework.

9.2 Extension to elasticity

As we have mentioned in the introduction, in this report we have dealt essentially with the propagation of sound in fluids. The question naturally arises whether one could use the same type of approach in dealing with sound propagation in solids. The possibility to build an analogue gravity based on acoustics in solids is suggested by two main considerations. The first is that we know that the analogue gravity paradigm only requires that the key equations for the model we intend to map to a relativistic dynamics must be hyperbolic, which is indeed the case of acoustic phenomena in solids. The second is that there are a number of approaches to general relativity which are based on elasticity, like e.g. Sakharov's induced gravity [48, 49]. Therefore, an interesting further development of the research pursued in this report would be the attempt to develop an analogue elasticity and, consequently, an analogue transformation elasticity.

From a more technological point of view, despite the number of proposals describing devices based on transformation-based solutions in the elasticity domain, so far only the practical realization of an elastic cloak for bending waves has been reported [50]. This experimental work was based on a theoretical proposal by Farhat and coworkers [51].

The possibility of designing elastic metamaterials would constitute an important breakthrough in the field of metamaterials with repercussion in many different technological sectors. Specifically in the space sector, it will open the way to metamaterials capable of making the launching phase considerably less stressful for the payload structure, and open the way to sonic exploration of planets and asteroids, with obvious advantages for geological assessments and mining.

9.3 Experiment phase

The fact that the metafluid properties required to implement the devices derived from our new approach to transformation acoustics are more natural and easy to synthesize that those of the standard approach, will certainly facilitate to a great extent the experimental realization of a variety of devices. A very interesting example would be the experimental demonstration of a three-dimensional acoustic Eaton lens, which is a perfect retroreflector. The optical version of this kind of device has important applications in space[9], communications, the enhancement of the radar signature of ships, etc.

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