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DIRECT MULTIPHASE OPTIMISATION OF MULTIOBJECTIVE TRAJECTORY DESIGN PROBLEMS

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INTRODUCTION

Some trajectory design problems require the optimization of more than one objective function. The quantities that are to be optimized can be competing or in accordance. An example of the first case is a spacecraft that has to reach one or more targets minimizing at the same time, or at different instants of time, the mass of propellant and the time to reach the target. An example of the second case is represented by two space vehicles flying in formation that have both to minimize the propellant mass to maintain their relative position and velocity.

It can even happen that the same quantity must be minimized by a player, as a pursuing spacecraft, and maximized by an other player, as a target spacecraft. This case is well known as a zero-sum two-player differential game[1] where the cost function must be maximized and minimized at the same time and is often associated to air combat problems where a pursuer has to intercept a target in minimum time.

All these problems can be classified as multiobjective optimization problems, in particular they can be divided into three categories depending on the objective function:

- competing objectives when one of the cost functions causes an increase of the others, if minimized
- concurrent objectives when the same cost function or different ones must be either minimized or maximized but by different players and with respect to different sets of controls;
- zero-sum mini-max problems when the same objective function must be maximized and minimized at the same time by different players

Standard direct approaches to trajectory optimization can not solve these classes of problems because they generally uses nonlinear programming solvers that can only maximize or minimize a single cost function.

Recently Conway[2] has proposed a semi-direct (semi-DCNLP) approach to the solution of pursuit/evasion problems. In this paper a fully direct approach is proposed which transcribes the original optimal control problem (or differential game) using a multiphase DFET[3] approach and solves the resulting nonlinear multiobjective programming problem. Each objective and each player are associated to a different phase, all the phases are then assembled together either sequentially or in parallel. In the case of a formation flight or of a pursuer and a target phases are solved in parallel sharing the same time domain..

The approach has been tested on three different sample cases, one for each class, finding efficiently a solution without resorting to adjoint variables and equations.

PROBLEM STATEMENT

The problem is to find the optimal policy for several players that have to reach one or more targets or objectives at the same time. Virtually the same holds for a one player that has to reach multiple objectives at the same time or at different instant of times but in the same framework.

If two players are considered the problem is to find the best policy that can satisfy the requirements of both players.

From an optimal control theory point of view this can be seen as a differential game in which two players have conflicting interests an the optimal policy must satisfy both requests. However, this can be seen even as a multiple objective problem in which the best policy, followed by the two players must optimize the objective function of each player at the same time.

From a game-theoretic point of view the problem can be formalized in the following way: find the optimal control or best policy u^o and v^o such that

$$J(u^o, v) \le J(u^o, v^o) \le J(u, v^o) \tag{1}$$

subject to

$$\begin{aligned} \dot{x} &= f(x,u,v,t) \\ g(x,u,v,t) &\ge 0 \end{aligned} \tag{2}$$

and boundary conditions

$$\phi(x, t_f) \ge 0 \tag{3}$$

According to optimal control theory the solution to the problem is obtained from:

$$H = L + \lambda^T f \tag{4}$$

where necessary conditions for optimality are therefore obtained as:

$$\dot{\lambda}^T = -H_x, \qquad \lambda^T(\mathrm{t_f}) = \Phi_{x(t_f)} \tag{5}$$

$$H_u = 0, \quad H_v = 0 \tag{6}$$

and the following second order conditions must hold:

$$H^o_{uu} \ge 0, \quad H^o_{vv} \le 0 \tag{7}$$

From a multiple objective optimization (MO) point of view the problem can be stated as: find the optimal control or best policy u^o and v^o such that

$$\min \qquad \mathbf{J}(u,v) = [J_1, \cdots, J_i, \cdots, J_M]^T \tag{8}$$

subject to a set of constraints:

$$\mathbf{c}(x,u,v) \ge 0 \tag{9}$$

where the vector **J** contains all conflicting interests and vector **c** all possible constraints.

The idea therefore is to solve, instead of a mini-max game-theoretic problem, a multiple objective optimization problem in which the policy of each player optimizes a different objective or criterion. In this sense the MO problem respects the requirement of separability expressed by the second-order conditions of optimal control theory.

OPTIMISATION APPROACH

In order to solve the problem expressed by (1),(2) and (3) in terms of problem (8) and (9) a multiphase direct transcription approach has been employed and the resulting multiple objective NLP problem has been either solved building an equivalent scalar objective function with all the objectives weighted in a proper way or with a trade-off approach.

Multiphase Approach

A general trajectory design problem can be decomposed in M phases[4], each one characterized by a time domain D^J , with $j=1,..,M$, a set of m dynamic variables $\mathbf{x}$, a set of n control variables $\mathbf{u}$ and a set of l parameters $\mathbf{p}$ Furthermore, each phase j may have an objective function

$$J^j = \phi^j(\mathbf{x}_0^b, \mathbf{x}_f^b, t_f, \mathbf{p}) + \int_{t_i}^{t_f} L^j(\mathbf{x}, \mathbf{u}, \mathbf{p})dt \tag{10}$$

a set of dynamic equations

$$\dot{\mathbf{x}} - \mathbf{F}^j(\mathbf{x}, \mathbf{u}, \mathbf{p}, t) = 0 \tag{11}$$

a set of algebraic constraints on states and controls

$$\mathbf{G}^j(\mathbf{x}, \mathbf{u}, \mathbf{p}, t) \geq \mathbf{0} \tag{12}$$

and a set of boundary constraints

$$\psi^j(\mathbf{x}_0^b, \mathbf{x}_f^b, \mathbf{p}, t)\Big|_{t_0}^{t_f} \geq 0 \tag{13}$$

Among boundary constraints a set of inter-phase link constraints exist that are used to assemble all phases together

$$\psi^j(\mathbf{x}_j^b, \mathbf{x}_{j-1}^b, \mathbf{p}, t) \geq 0 \tag{14}$$

The resulting multiple objective problem can be treated in different ways according to the original optimization problem. If the players are cooperative, a global objective function expressed as a weighted sum of all the objective functions can be formed:

$$J = \sum_{i=1}^{M} \beta_i J_i \tag{15}$$

Ones the objective function are suitably blended, the Pareto's optimal region of the associated multiobjective optimization problem[5] can be obtained. The point closest to the utopia point, i.e. the vector formed by the individual minima of all the objective functions, of the criterion space is taken as the solution.

On the other hand, if the players are non-cooperative, a weighted sum leads to non-optimal results and a trade-off approach has to be used: for each phase, the objective functions of the other phases are seen as constraints.

$$\begin{aligned} &\min \;\; J_i \\ &J_j \leq 0 \qquad \mathrm{j = 1,..,M \;\; and \;\; j \neq i} \end{aligned} \tag{16}$$

Transcription Method

The time domain $D(t_0,t_f)\subset\Re$ relative to each phase j can be further decomposed into N finite time elements $D^j = \bigcup_{i=1}^{N} D_i^j(t_{i-1}, t_i)$ and, on each time element D^j_i, states and controls $[\mathbf{x,u}]$ can be parameterized as follows:

$$\begin{Bmatrix} \mathbf{x} \\ \mathbf{u} \end{Bmatrix} = \sum_{s=1}^{p} f_s(t) \begin{Bmatrix} \mathbf{x}_s \\ \mathbf{u}_s \end{Bmatrix} \tag{17}$$

where the basis functions f_s are chosen within the space of polynomials of order *p-1*:

$$f_s \in P^{p-1}(D_i^j) \tag{18}$$

Therefore in general a finite element is defined by a sub-domain D^j_i, and by a sub-set of parameters $[\mathbf{x}_s, \mathbf{u}_s, \mathbf{p}]$. A group of finite elements forms a phase and a group of phases forms the original optimization problem. Notice that additional parameters **p** may occur in all constraint equations depending on their function in the optimization problem. Furthermore it should be noticed that each phase can be grouped in sequence or in parallel with the other phases depending on its time domain and on the inter-phase link constraints that pass information among phases. Thus two phases can share the same time domain but have different parameterizations.

Now taking a general phase, in order to integrate differential constraints (11),on each finite element *i*, differential equations are transcribed into a weighted residual form considering boundary conditions of the weak type:

$$\int_{t_i}^{t_{i+1}} \left\{ \dot{\mathbf{w}}^T \mathbf{x} + \mathbf{w}^T \mathbf{F}^j \right\} dt - \mathbf{w}_{i+1}^T \mathbf{x}_{i+1}^b + \mathbf{w}_i^T \mathbf{x}_i^b = 0 \qquad i = 1, ..., N-1 \tag{19}$$

where $\mathbf{w}(t)$ are generalized weight (or test) functions defined as:

$$\mathbf{w} = \sum_{s=1}^{p+1} g_s(t) \mathbf{w}_s \tag{20}$$

where g_s are taken within the space of polynomials of order p:

$$g_s \in P^p(D_i^j) \tag{21}$$

Now the problem is to find the vector $\mathbf{x}_s \in \Re^{p*m}$, the vector $\mathbf{u}_s \in \Re^{p*n}$, the vector $\mathbf{p} \in \Re^l$ and $\mathbf{x}^b_f$ and $\mathbf{x}^b_0 \in \Re^m$ that satisfy variational equation (19) along with algebraic and boundary constraints:

$$\mathbf{G}^j(\mathbf{x}, \mathbf{u}, \mathbf{p}, t) \geq 0 \tag{22}$$

$$\psi^j(\mathbf{x}_0^b, \mathbf{x}_f^b, \mathbf{p}, t)\Big|_{t_0}^{t_f} = 0 \tag{23}$$

where quantities $\mathbf{x}_s$, and $\mathbf{u}_s$ are called internal node values, while $\mathbf{x}^b_f$, $\mathbf{x}^b_0$ are called boundary values. Notice that generally the order p of the polynomials can be different for states and controls. In a more general way the domain D^j could be decomposed as a union of smooth images of the reference time interval [-1,1] where a reference parameter τ is defined as:

$$\tau = 2\frac{t - t_{i-1/2}}{t_i - t_{i-1}} = 2\frac{t - t_{i-1/2}}{\Delta t_i} \tag{24}$$

Polynomials f_s and g_s are constructed using Lagrangian interpolants associated with internal Gauss-type nodes. Generally speaking if $\{\xi_s\}^p_{s=1}$ are the set of Gauss points on the reference interval [-1,1], $f_s(\tau)$ will be the Lagrangian interpolating polynomial vanishing at all Gauss points except at ξ_s where it equals one.

Each integral of the continuous forms (19) and (10) is then replaced by a q-points Gauss quadrature sum, where q is taken equal to p. Therefore the objective function (10) becomes a sum of N Gauss quadrature formulas:

$$J^j = \phi^j(\mathbf{x}_0^b, \mathbf{x}_f^b, t_f) + \sum_{i=1}^{N} \sum_{k=1}^{q} \sigma_k L^j{}_k \frac{\Delta t_i}{2} \tag{25}$$

while integral (19) is split into N integrals of the form:

$$\sum_{k=1}^{q} \sigma_k \left[\dot{\mathbf{w}}_k(\tau_k)^T \mathbf{x}(\tau_k) + \mathbf{w}_k(\tau_k)^T \mathbf{F}^j{}_k \frac{\Delta t_i}{2} \right] - \mathbf{w}_{p+1}^T \mathbf{x}_{i+1}^b + \mathbf{w}_1^T \mathbf{x}_i^b = 0 \qquad i=1,\dots,N-1 \tag{26}$$

where σ_k are Gauss weights and parameters $\mathbf{x}^b_{i-1}$ and $\mathbf{x}^b_i$ are boundary values at the beginning and at the end of each element. For sake of simplicity, the following notation has been introduced:

$$L^j{}_k = L^j(\mathbf{x}_s f_s(\tau_k), \mathbf{u}_s f_s(\tau_k), \mathbf{p}, \tau_k); \qquad \mathbf{F}^j{}_k = \mathbf{F}^j(\mathbf{x}_s f_s(\tau_k), \mathbf{u}_s f_s(\tau_k), \mathbf{p}, \tau_k) \tag{27}$$

Here controls are parameterized using the same set of points used for integration while states are always collocated on Gauss-Lobatto nodes. Numerical quadrature of the integral Eq. (19) and integral (10) can be then performed either by Gauss Lobatto rule or by Gauss-Legendre rule. The former choice of quadrature formulas collocates controls on the same set of nodes as states while the latter collocates controls on a different set. The advantage of the latter is the higher integration order which allows a lower number of collocation nodes. Whatever f_s and g_s are, the linear part of Eq. (26) can be always integrated only once before the optimization process begins. Now Eq. (26) must be satisfied for every arbitrary value of virtual quantity $\mathbf{w}_k$, as a consequence each element equation is developed into $p+1$ equations:

$$\sum_{k=1}^{q}\sigma_k \mathbf{F}^j{}_k \frac{\Delta t_i}{2}\begin{Bmatrix} g_1(\tau_k) \\ \vdots \\ g_{p+1}(\tau_k) \end{Bmatrix} + \begin{bmatrix} \sum_{k=1}^{q}\sigma_k \dot{g}_1(\tau_k) f_1(\tau_k) & \cdots & \sum_{k=1}^{q}\sigma_k \dot{g}_1(\tau_k) f_p(\tau_k) \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{q}\sigma_k \dot{g}_{p+1}(\tau_k) f(\tau_k)_1 & \cdots & \sum_{k=1}^{q}\sigma_k \dot{g}_{p+1}(\tau_k) f_p(\tau_k) \end{bmatrix} \begin{Bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_p \end{Bmatrix} = \begin{bmatrix} -\mathbf{x}^b_i \\ 0 \\ \mathbf{x}^b_{i+1} \end{bmatrix} \tag{28}$$

System of Eqs. (28) is written for each element, all the elements are then assembled matching the final boundary node of one element to the initial one of the next element. For continuous solution, in order to preserve the continuity of the states, at matching points, the following condition must hold:

$$\mathbf{x}^b_i = \mathbf{x}^b_{i+1} \qquad i=1,\ldots,N\text{-}2 \tag{29}$$

Thus all the boundary quantities (29) cancel one another except for those at the initial and final times. Algebraic constraint equation (22) can be collocated directly at Gauss nodal points:

$$\mathbf{G}^j{}_s(\mathbf{x}_s(\xi_s), \mathbf{u}_s(\xi_s), \mathbf{p}, \xi_s) \geq 0 \tag{30}$$

The resulting set of non-linear algebraic equations, assembling all the phases, along with discretized objective function (25) can be seen as a general non-linear programming problem (NLP) of the form:

$$\min \quad \mathrm{J}(\mathbf{y}) \tag{31}$$

subject to

$$\begin{aligned} &\mathbf{c}(\mathbf{y}) \geq 0 \\ &\mathbf{b}_l \leq \mathbf{y} \leq \mathbf{b}_u \end{aligned} \tag{32}$$

where, $\mathbf{y}$ is the vector of NLP variables, $\mathrm{J}(\mathbf{y})$ the objective function to be minimized, $\mathbf{c}(\mathbf{y})$ a vector of non-linear constraints and $\mathbf{b}_l$ and $\mathbf{b}_u$ respectively lower and upper bounds on NLP variables. The $N*(p+1)*n$ algebraic Eqs. (28) taken for each phase, along with system (30), represent the $\mathbf{c}(\mathbf{y})$ constraint of the nonlinear problem while $\mathbf{y}=[\mathbf{x}_s,\mathbf{u}_s,\mathbf{x}^b{}_0,\mathbf{x}^b{}_f,t_0,t_f,\mathbf{p}]$ the NLP variables.

SAMPLE PROBLEMS

Some sample problems are presented to show how the multiphase approach can be used to solve problems with multiple players and multiple objectives. Each problem has been taken simple enough to allow an easy understanding of all the issues related to the solution of this kind of problems with a fully direct method and to allow a comparison with the results coming from optimal control theory.

Competing Objective Functions

A simple case of competing objective function is represented by two spacecrafts, which have to reach a given target in minimum time flying in formation but one of them, for some reasons, can not minimize the time because of a lack of propellant.
One is therefore minimizing time and the other is minimizing propellant mass and the two policies are, of course, in conflict. The optimal policy therefore would be a compromise between each policy of each single player.

Optimal Control Formulation

The dynamics of the two spacecraft is governed by the following simple system of differential equations:

$$\begin{bmatrix} \dot{v}_1 \\ \dot{h}_1 \\ \dot{m}_1 \\ \dot{v}_2 \\ \dot{h}_2 \\ \dot{m}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{F_1}{m_1} \\ v_1 \\ -c_1 F_1 \\ \dfrac{F_2}{m_2} \\ v_2 \\ -c_2 F_2 \end{bmatrix} \tag{33}$$

the two objective functions can be blended in the following way leading to a unique minimization problem :

$$\min J = -m(t_f) + \beta \int_0^{t_f} dt \tag{34}$$

A second order equality constraint on state is then added

$$h_1 - h_2 = k \tag{35}$$

to state that the two spacecraft must fly in formation and are not independent one from the other., and the thrust provided by the two spacecraft is limited.

$$\begin{gathered} 0 \le F_1 \le F_{\max} \\ 0 \le F_2 \le F_{\max} \end{gathered} \tag{36}$$

According to control theory the second time derivative of constraint (35) must be taken, therefore the following relationship between the controls of the two spacecraft must hold:

$$\frac{F_1}{m_1} - \frac{F_2}{m_2} = 0 \tag{37}$$

The same constraint yields, for an optimal solution, the following relationship between the velocities:

$$v_1 = v_2 \tag{38}$$

the Hamiltonian to the problem is therefore:

$$H = 1 + \lambda_{v1} F_1 / m_1 + \lambda_{x1} v_1 - \lambda_{m1} c_1 F_1 + \lambda_{v2} F_2 / m_2 + \lambda_{x2} v_2 - \lambda_{m2} c_2 F_2 + \mu \left(\frac{F_1}{m_1} - \frac{F_2}{m_2} \right) \quad (39)$$

and in order to be minimal the following quantities must be minimized with respect to the two controls:

$$\left(\frac{\lambda_{v1}}{m_1} - \lambda_{m1} c_1 + \mu \frac{1}{m_1} \right) F_1 \quad (40)$$

$$\left(\frac{\lambda_{v2}}{m_2} - \lambda_{m2} c_2 - \mu \frac{1}{m_2} \right) F_2 \quad (41)$$

Direct Multiphase Multiobjective Formulation

If the problem is solved using a direct multiple objective approach the dynamics of the two spacecrafts is split in two phases and each dynamic system is assigned to a different phase:

$$\begin{bmatrix} \dot{v}_1 \\ \dot{h}_1 \\ \dot{m}_1 \end{bmatrix} = \begin{bmatrix} \frac{F_1}{m_1} \\ v_1 \\ -c_1 F_1 \end{bmatrix}, \qquad \begin{bmatrix} \dot{v}_2 \\ \dot{h}_2 \\ \dot{m}_2 \end{bmatrix} = \begin{bmatrix} \frac{F_2}{m_2} \\ v_2 \\ -c_2 F_2 \end{bmatrix} \quad (42)$$

the time domain of each phase must be linked together in order to have interdependent terminal conditions, therefore:

$$t_{1f} - t_{2f} = 0 \quad (43)$$

the other relevant constraints remain the same as for optimal control formulation:

$$h_1 - h_2 = k \quad (44)$$

$$\begin{aligned} 0 \le F_1 \le F_{1\max} \\ 0 \le F_2 \le F_{2\max} \end{aligned} \quad (45)$$

It should be noticed that in this case the two spacecrafts have conflicting interest, represented by the two objective functions, but are collaborating to reach the best result possible. Therefore the two objective functions can be combined to form a unique scalar quantity:

$$J = J_1 + \beta J_2 \quad (46)$$

Now, depending on the value of the weight β, the optimal solution for the two objective functions correspond to a point in the Pareto's optimal region displayed in Fig. 1. It is now easy, in two dimensions, to rank all the optimal solutions found. The best one is the closest to the utopia point O and corresponds to the minimum value for each one of the objective functions if constrained by the other one.

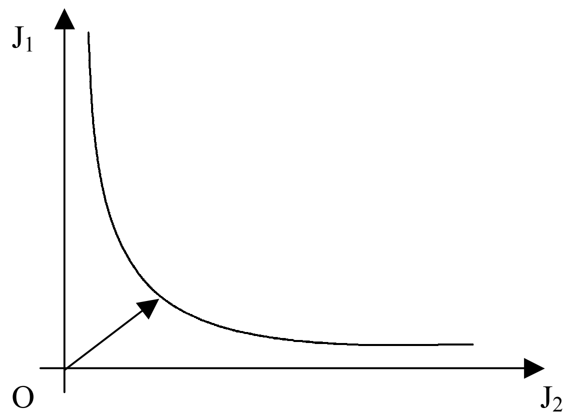

Figure 1. Pareto's optimal region

In this way for each value of β a standard NLP solver can solve the problem. The Pareto optimal curve for the problem analyzed here has been represented in Fig. 2. Taking the best couple of values, corresponding to the point closest to the utopia point, as mentioned before, the best compromise between the two policies of the two collaborating players can be obtained. The corresponding solution, yielded by the direct multiphase multiobjective approach (DMM) has been plotted in Figs. from 3 to 6 along with solution coming from optimal control theory (Indirect).

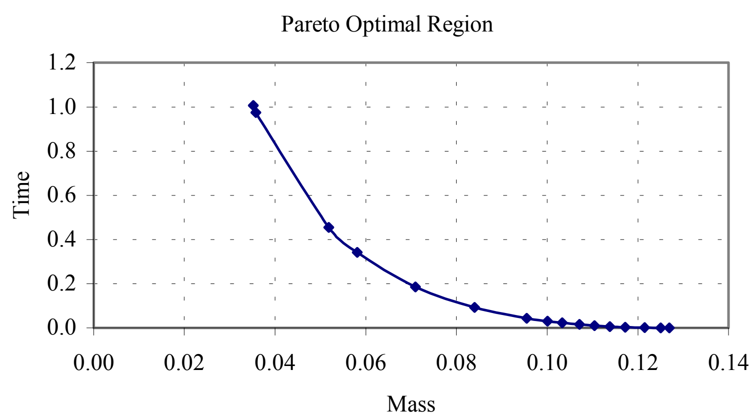

Figure 2. Pareto's optimal region showing all optimal solutions in terms of time and mass

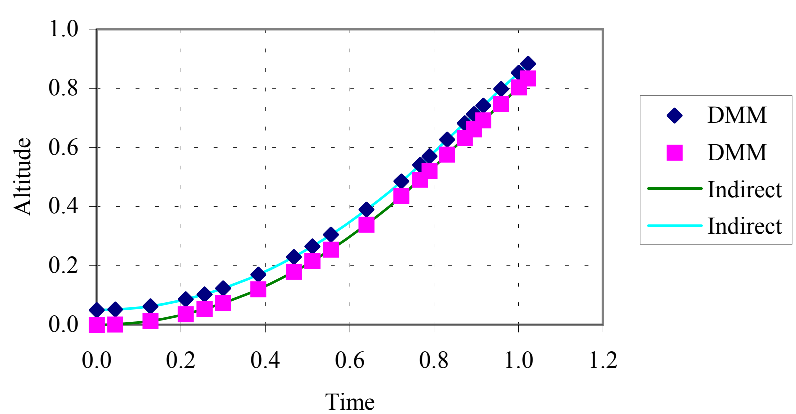

Figure 3. Altitude time history: comparison between direct and indirect solution

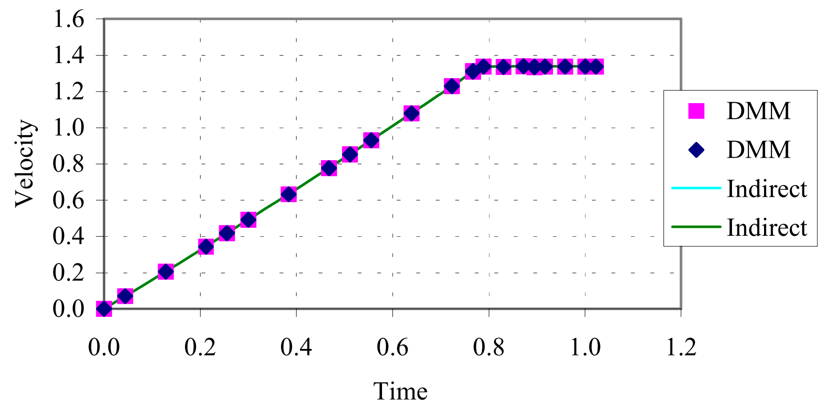

Figure 4. Velocity time history: comparison between direct and indirect solution.

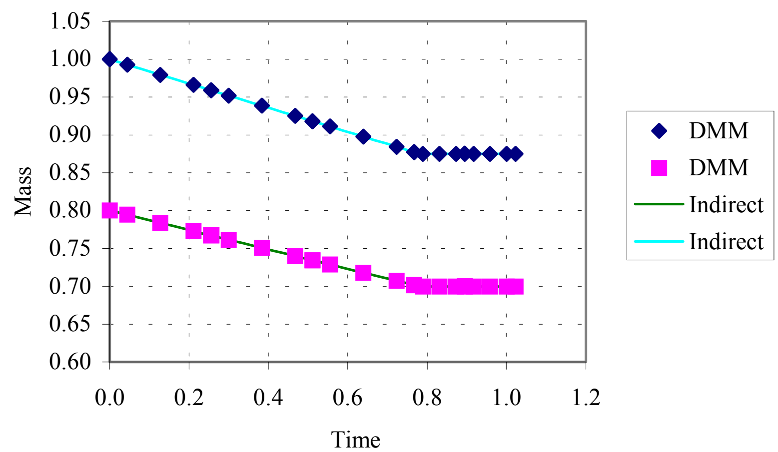

Figure 5. Propellant consumption: comparison between direct and indirect solution

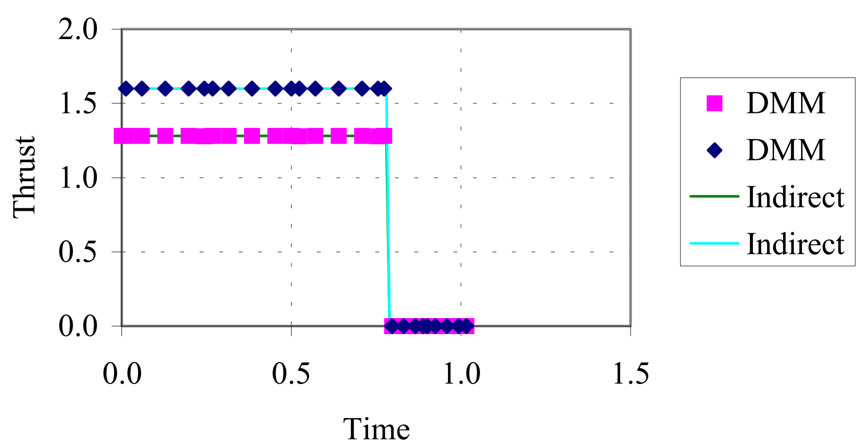

Figure 6. Optimal control with respect to time

Concurrent Objective Functions

The second case analyzed is an example of two spacecraft, which have to minimize the same objective function. This is typical for two spacecrafts flying in formation that wants to minimize the control to reach a given target while maintaining their relative distance.
The problem can be easily translated into a single objective function problem with the dynamics of the two spacecraft as differential constraints.

Optimal Control Formulation

The general formulation of the problem can be stated using optimal control theory in the following way: minimize the integral of the square of the control of the two spacecrafts:

$$J = \int_{t_0}^{t_f} \frac{1}{2}(u_1^2 + u_2^2)dt \tag{47}$$

subject to the dynamic system governing the motion of the two spacecrafts flying in formation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{v}_{x1} \\ \dot{v}_{y1} \\ \dot{x}_2 \\ \dot{y}_2 \\ \dot{v}_{x2} \\ \dot{v}_{y2} \end{bmatrix} = \begin{bmatrix} v_{x1} \\ v_{y1} \\ u_{x1} - {kx_1}/{r_1^3} \\ u_{y1} - {ky_1}/{r_1^3} \\ v_{x2} \\ v_{y2} \\ u_{x2} - {kx_2}/{r_2^3} \\ u_{y2} - {ky_2}/{r_2^3} \end{bmatrix} \tag{48}$$

where

$$\begin{gathered} r_1 = \sqrt{{x_1}^2 + {y_1}^2},\ r_2 = \sqrt{{x_2}^2 + {y_2}^2} \\ u_1 = \sqrt{u_{x1} + u_{y1}},\ u_2 = \sqrt{u_{x2} + u_{y2}} \end{gathered} \tag{49}$$

The two spacecrafts must maintain their relative position therefore an algebraic constraint has to be added on their path (see Fig.7):

$$r_1(t) - r_2(t) = \rho \tag{50}$$

If the second total time derivative of constraint (50) is taken:

$$(v_{x1} - v_{x2})^2 + (v_{y1} - v_{y2})^2 + (x_1 - x_2)\left(u_{x1} - \frac{kx_1}{r_1^2} - u_{x2} + \frac{kx_2}{r_2^2}\right) + (y_1 - y_2)\left(u_{y1} - \frac{ky_1}{r_1^2} - u_{y2} + \frac{ky_2}{r_2^2}\right) = 0 \tag{51}$$

the Hamiltonian for problem (47),(48) can be written as

$$H = \frac{1}{2}(u_1^2 + u_2^2) + \lambda_{x1} v_{x1} + \lambda_{y1} v_{y1} + \lambda_{v_{x1}} \left(u_{x1} - \frac{kx_1}{r_1^2} \right) + \lambda_{v_{y1}} \left(u_{y1} - \frac{ky_1}{r_1^2} \right) +$$
$$\lambda_{x2} v_{x2} + \lambda_{y2} v_{y2} + \lambda_{v_{x2}} \left(u_{x2} - \frac{kx_2}{r_2^2} \right) + \lambda_{v_{y2}} \left(u_{y2} - \frac{ky_2}{r_2^2} \right) +$$
$$2\mu \left[(v_{x1} - v_{x2})^2 + (v_{y1} - v_{y2})^2 + (x_1 - x_2) \left(u_{x1} - \frac{kx_1}{r_1^2} - u_{x2} + \frac{kx_2}{r_2^2} \right) + (y_1 - y_2) \left(u_{y1} - \frac{ky_1}{r_1^2} - u_{y2} + \frac{ky_2}{r_2^2} \right) \right] \tag{52}$$

and the optimality conditions for the control of the first spacecraft are:

$$\begin{aligned} \frac{\partial H}{\partial u_{x1}} &= u_{x1} + \lambda_{v_{x1}} + 2\mu(x_1 - x_2) = 0 \\ \frac{\partial H}{\partial u_{y1}} &= u_{y1} + \lambda_{v_{y1}} + 2\mu(y_1 - y_2) = 0 \end{aligned} \tag{53}$$

If the following final constraint on position is imposed:

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = c \tag{54}$$

terminal values for the adjoint variables are:

$$\begin{aligned} &\lambda_{x1}(t_f) = -\nu_1 \, {}^{y_1}\!/_{x_1^2}, \quad \lambda_{y1}(t_f) = \nu_1 \, {}^{1}\!/_{x_1} \\ &\lambda_{x2}(t_f) = -\nu_2 \, {}^{y_2}\!/_{x_2^2}, \quad \lambda_{y2}(t_f) = \nu_2 \, {}^{1}\!/_{x_2} \\ &\lambda_{v_{x1}}(t_f) = 0, \lambda_{v_{y1}}(t_f) = 0, \lambda_{v_{x2}}(t_f) = 0, \lambda_{v_{y2}}(t_f) = 0 \end{aligned} \tag{55}$$

from which the terminal conditions for the controls of the two spacecrafts are:

$$\begin{aligned} u_{x2}(t_f) &= -u_{x1}(t_f); \\ u_{y2}(t_f) &= -u_{y1}(t_f) \end{aligned} \tag{56}$$

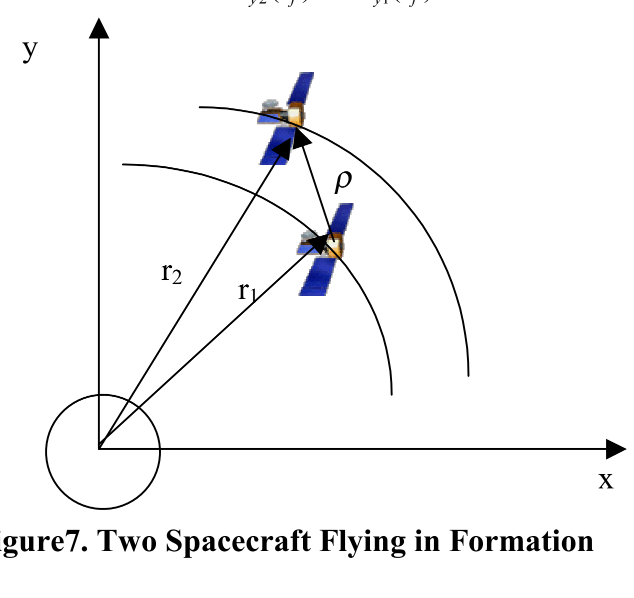

Figure7. Two Spacecraft Flying in Formation

Direct Multiphase Multiobjective Formulation

If a multiphase-multiobjective approach is used, the problem is split in two phases, one for each spacecraft. The first phase contains the dynamics of the first spacecraft and the objective function that has to be minimized by the control associated to the first phase:

$$J_1 = \int_{t_0}^{t_f} \frac{1}{2} u_1^2 \tag{57}$$

The second phase contains the dynamics of the second spacecraft the objective function that has to be minimized by the control associated to the second phase:

$$J_2 = \int_{t_0}^{t_f} \frac{1}{2} u_2^2 \tag{58}$$

and constraint (50) on the relative position of the two spacecrafts which must be constant along the path. Constraint (50) however does not provide all the necessary information required for optimality and therefore constraint (51) is explicitly enforced.
The two phases are then solved in parallel and the interaction between the two spacecrafts is maintained through constraint (51) and through the additional constraint:

$$t_{1f} - t_{2f} = 0 \tag{59}$$

that forces time domain of phase one to be equal to time domain of phase two.
The present formulation leads naturally to the solution of the problem on a parallel machine with two processes, one for each phase, communicating through paths constraint (51) and boundary constraint (59).

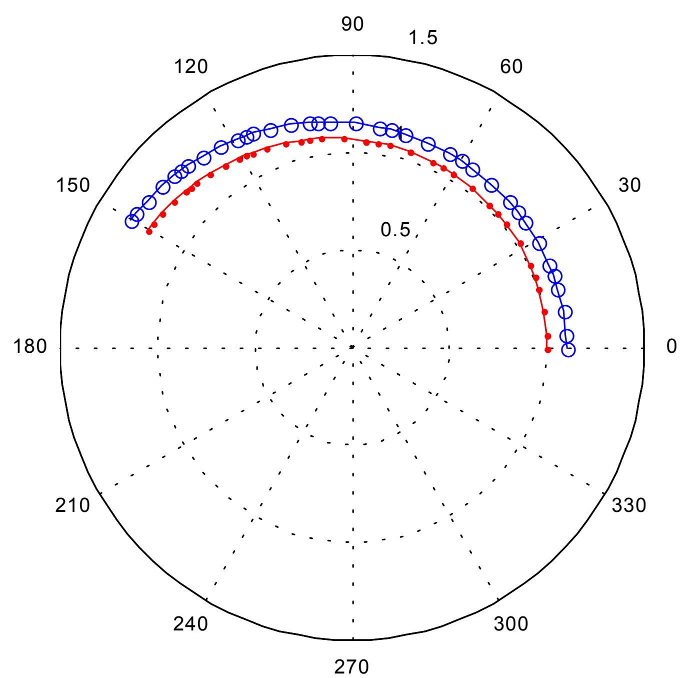

Figure 8. Spacecrafts flying in formation

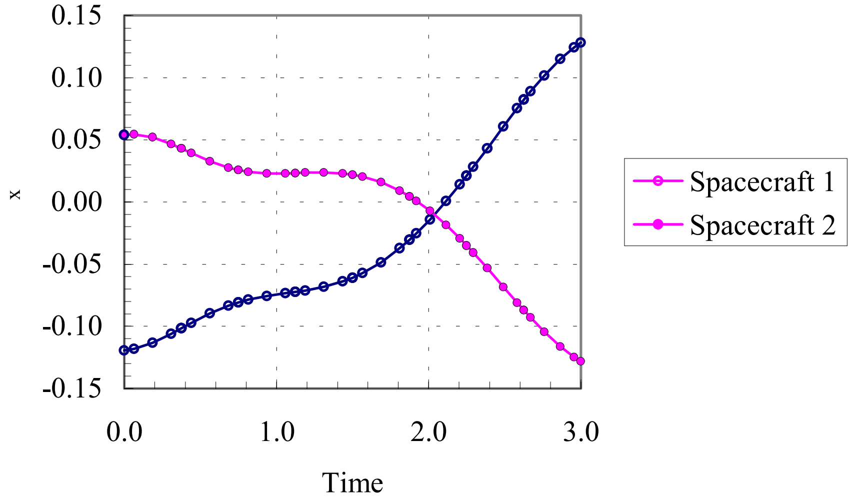

Figure 9. Thrust along x axis

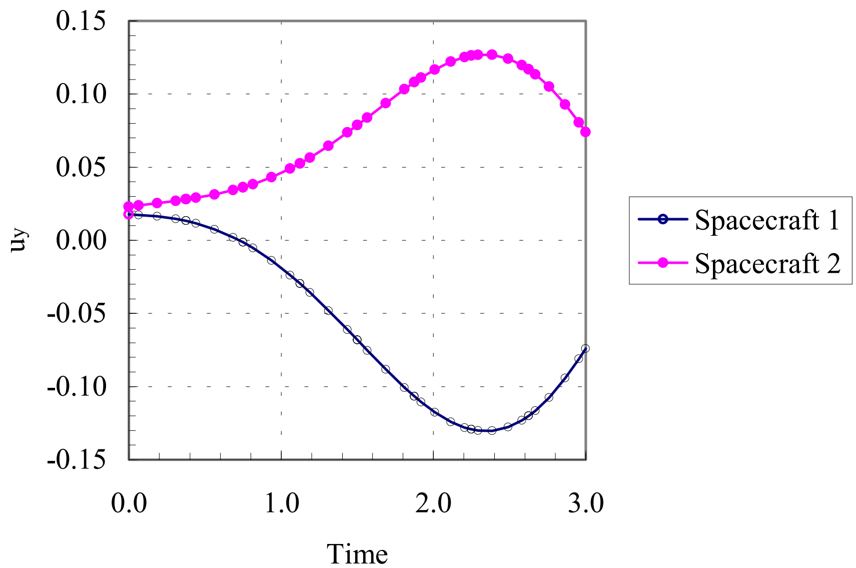

Figure 10.Thrust along the y axis

In Fig. 8 the resulting trajectory has been reported for a normalized $k=1$, an initial position vector for spacecraft 1 equal to [1,1] and for spacecraft 2 [1.1,1] and a normalized flight time equal to 3 and a ρ=0.1. Final condition, for both spacecrafts, is to reach an angular position of 150°. In Figs. 9 and 10 the thrust control law along the x and y axis have been plot respectively**.** It should be noticed how the terminal conditions for the controls got from MDFET respect Eq. 56**.**

Zero-Sum Mini-Max Problems

In a simple 1-dimensional pursuit-evader game[1], the pursuer's control is his acceleration a_p and the evader control is his acceleration a_e. The normal time of closest approach is t_f and the displacement between the pursuer and the evader is y:

$$\begin{cases}\dot{x}_p = v_p \\ \dot{v}_p = a_p\end{cases}, \quad \begin{cases}\dot{x}_e = v_e \\ \dot{v}_e = a_e\end{cases} \tag{60}$$

given a set of initial conditions for both players, with the evader starting ahead of the pursuer, the pursuer whishes to minimize the time to bring the terminal miss y(t_f) to zero, whereas the evader wishes to maximize it, so the performance index may be taken as:

$$J = t_f \tag{61}$$

with the constraint:

$$y(t_f) = x_p(t_f) - x_e(t_f) = 0 \tag{62}$$

the acceleration of the two player are of course limited and the one of the pursuer is bigger than that of the evader.

Optimal Control Formulation

According to optimal control theory the solution to this problem proceeds by first forming the Hamiltonian:

$$H = 1 + \lambda_{xp} v_p + \lambda_{vp} a_p + \lambda_{xe} v_e + \lambda_{ve} a_e \tag{63}$$

The adjoint equations are then:

$$\begin{cases}\dot{\lambda}_{xp} = 0 \\ \dot{\lambda}_{vp} = -\lambda_{xp}\end{cases}, \quad \begin{cases}\dot{\lambda}_{xe} = 0 \\ \dot{\lambda}_{ve} = -\lambda_{xe}\end{cases}, \quad \begin{cases}\lambda_{\mathrm{vp}}(t_f) = 0 \\ \lambda_{\mathrm{xp}}(t_f) = y(t_f)\end{cases}, \quad \begin{cases}\lambda_{\mathrm{ve}}(t_f) = 0 \\ \lambda_{\mathrm{xe}}(t_f) = -y(t_f)\end{cases} \tag{64}$$

and the resulting optimal policy for the two player can be expressed through the following optimality conditions:

$$\begin{aligned} a_p &= -a_{p\max} \operatorname{sgn} \lambda_{vp} \\ a_e &= -a_{e\max} \operatorname{sgn} \lambda_{ve} \end{aligned} \tag{65}$$

and due to equations (64) the sign of the adjoints λ_{vp} and λ_{ve} is constant. The mini-max solution, therefore, corresponds to a maximum use of the control for both players.

Direct Multiphase Multiobjective Formulation

If direct transcription is applied with multiobjective, multiphase formulation the objective function for the pursuer is the time required to reach the evader, while the evader has to maximize the terminal miss y. The two players are not collaborative and therefore the two objective functions cannot be blended to form a unique scalar objective to be minimized.

The corresponding multiple objective problem is then treated with a trade-off approach. The original problem is split in the following two one-side problems:

$$\begin{array}{lcl} \min\ t_{pf} & & \min -\dfrac{1}{2}(x_e(t_{ef}) - x_p(t_{pf}))^2 \\ \text{subject to} & \text{and} & \text{subject to} \\ x_p(t_{pf}) - x_e(t_{ef}) \le k & & t_{pf} = t_{ef} \end{array} \tag{66}$$

Phase 1, the pursuer, has to minimise the time to reach a determined final miss equal to a constant k while Phase 2, the evader, has to maximise the final miss at fixed time. The two objective k and t_{pf} must then be minimized. The value obtained from phase 2 can then be used to solve the problem of phase 1 and the value obtained from the solution of phase 1 is used to solve phase 2.

The minimum value for k is of course zero and this is the maximum value that can be obtained by the evader (the pursuer must reach the evader in finite time). On the other hand the minimum time required by the pursuer to reach the target correspond to the optimal solution.

In Fig.11 the result obtained with DMM has been compared with optimal control solution (Indirect) for the 1-dimesnional pursuer-evader game with the two player starting with the same position. The evader has an initial velocity of 0.1 and a maximum acceleration of 0.01 while the pursuer has an initial velocity of 0 and a maximum acceleration of 0.05.

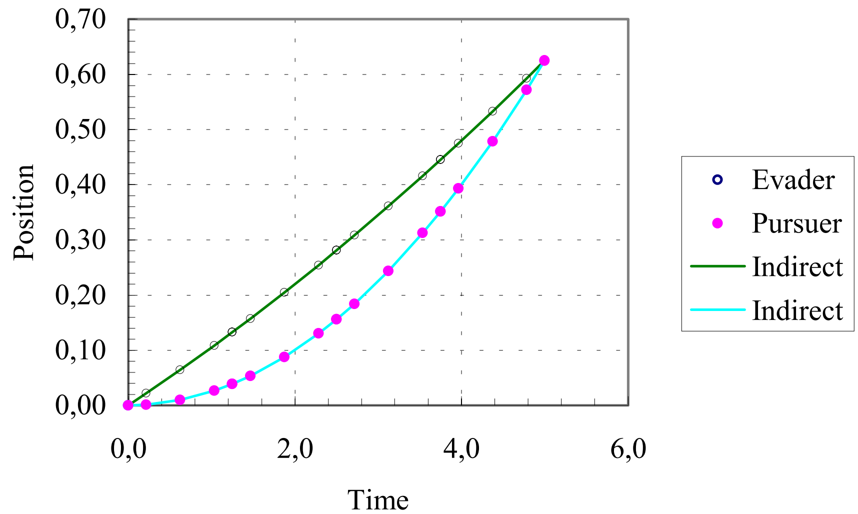

Figure 11. Solution for the 1-dimensional pursuer-evader problem.

Recently, a simple pursuer-evader problem has been presented by Conway[2]: a target spacecraft must be intercepted by a pursuer spacecraft in minimum finite time (see Fig.12). The problem, solved by Conway using semi-DCNLP approach, is here solved with the proposed Direct Multiphase Multiobjective approach (DMM).

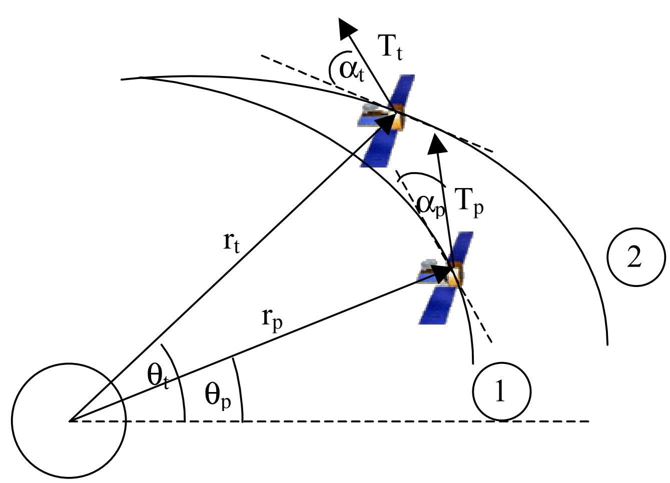

Figure 12. Spacecraft Interception of Optimally Evasive Target

Optimal Control Formulation

In the optimal control version the pursuer wishes to minimize the time to intercept the target, that is to say to reduce the terminal miss to zero in minimum time:

$$J = t_f \tag{67}$$

on the other hand the target whishes to maximize the time.

The dynamics of the two spacecrafts is given by the following system of differential equations

$$\begin{bmatrix} \dot{r}_t \\ \dot{\theta}_t \\ \dot{v}_{rt} \\ \dot{v}_{\theta t} \\ \dot{r}_p \\ \dot{\theta}_p \\ \dot{v}_{rp} \\ \dot{v}_{\theta p} \end{bmatrix} = \begin{bmatrix} v_{rt} \\ v_{\theta t}/r_t \\ T_t \sin\alpha_t - (\mu - v_{\theta t}^2 r_t)/r_t^2 \\ T_t \cos\alpha_t - v_{rt} v_{\theta t}/r_t \\ v_{rp} \\ v_{\theta p}/r_p \\ T_p \sin\alpha_p - (\mu - v_{\theta p}^2 r_p)/r_p^2 \\ T_p \cos\alpha_p - v_{rp} v_{\theta p}/r_p \end{bmatrix} \tag{68}$$

and the final miss must be zero:

$$\begin{bmatrix} r_t(t_f) - r_p(t_f) \\ \theta_t(t_f) - \theta_p(t_f) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{69}$$

The solution proceeds by first forming the Hamiltonian:

$$\begin{aligned} H = 1 + \lambda_{rt} v_{rt} + \lambda_{\theta t} \frac{v_{\theta t}}{r_t} + \lambda_{v_{rt}} \left[T_t sin\alpha_t - \frac{(\mu - v_{\theta t}^2 r_t)}{r_t^2} \right] + \lambda_{v_{\theta t}} \left[T_t \cos\alpha_t - \frac{v_{rt} v_{\theta t}}{r_t} \right] + \\ \lambda_{rp} v_{rp} + \lambda_{\theta p} \frac{v_{\theta p}}{r_p} + \lambda_{v_{rp}} \left[T_p sin\alpha_p - \frac{(\mu - v_{\theta p}^2 r_p)}{r_p^2} \right] + \lambda_{v_{\theta p}} \left[T_p \cos\alpha_p - \frac{v_{rp} v_{\theta p}}{r_p} \right] \end{aligned} \tag{70}$$

and then by obtaining the optimal control law for both pursuer and target can be expressed in the following form:

$$\tan\alpha_t = \frac{\lambda_{v_{rt}}}{\lambda_{v_{\theta t}}} \qquad \text{and} \qquad \tan\alpha_p = \frac{\lambda_{v_{rp}}}{\lambda_{v_{\theta p}}} \tag{71}$$

Direct Multiphase Multiobjective Formulation

The problem is a zero-sum differential game with two non-collaborative players as before and cannot be translated, into a nonlinear programming problem with a single objective function. As stated by Conway, standard NLP solvers cannot handle multiple objectives but furthermore they cannot handle two objective functions that have zero-sum as in this case.

Therefore, as above, a trade-off approach is used to treat the resulting multiple objective problem.

One objective function is the time that the pursuer needs to reach the target:

$$J_1 = t_{fp} \tag{72}$$

and is associated to a first phase with the additional constraint:

$$\begin{bmatrix} r_p(t_f) - r_t(t_f) \\ \theta_p(t_f) - \theta_t(t_f) \end{bmatrix} = \Delta \tag{73}$$

The second phase is then associated to the target with the additional constraint stating that the two phases must share the same time domain:

$$t_{fp} - t_{ft} = 0 \tag{74}$$

The objective function associated to the target is the final miss:

$$J_2 = -\frac{1}{2}|\Delta|^2 \tag{75}$$

Even in this case, applying the same considerations of the previous 1-dimensional problem, the best policy for both is to use the maximum control to reach a zero miss (the objective of the target) in minimum time (the objective of the pursuer).

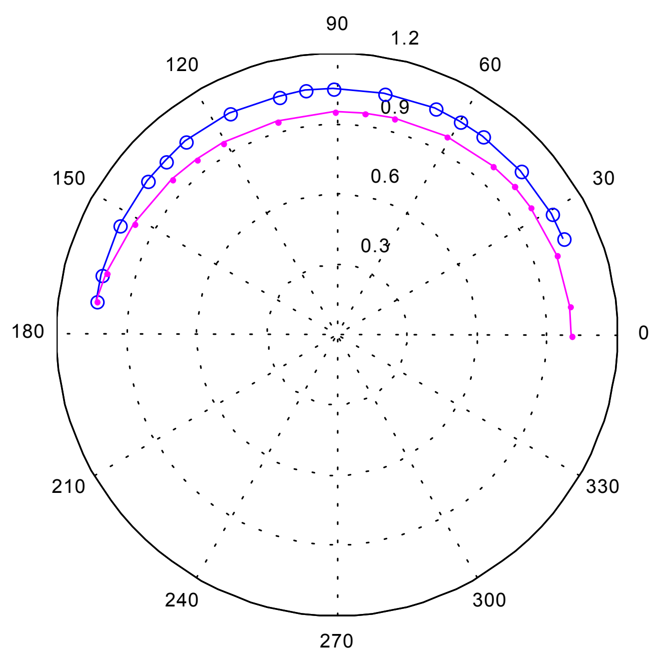

Figure 13.Target and Pursuer Trajectory

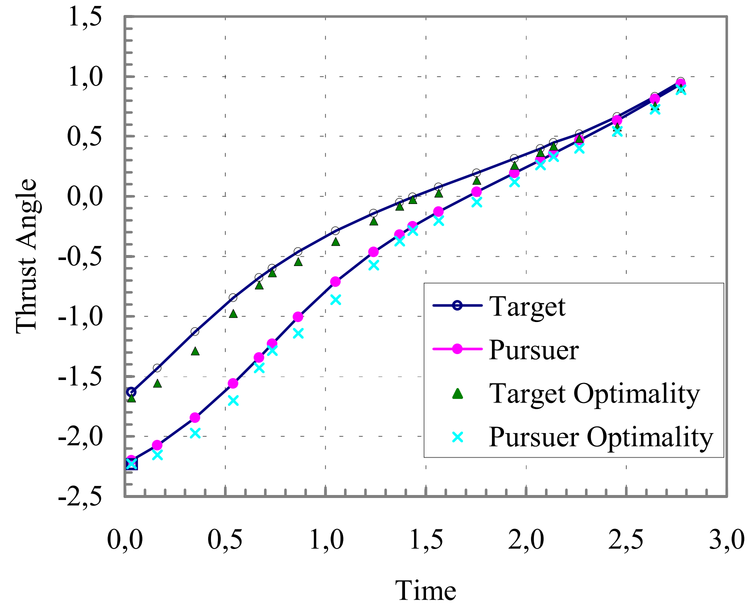

Figure 14. Thrust Pointing Angles

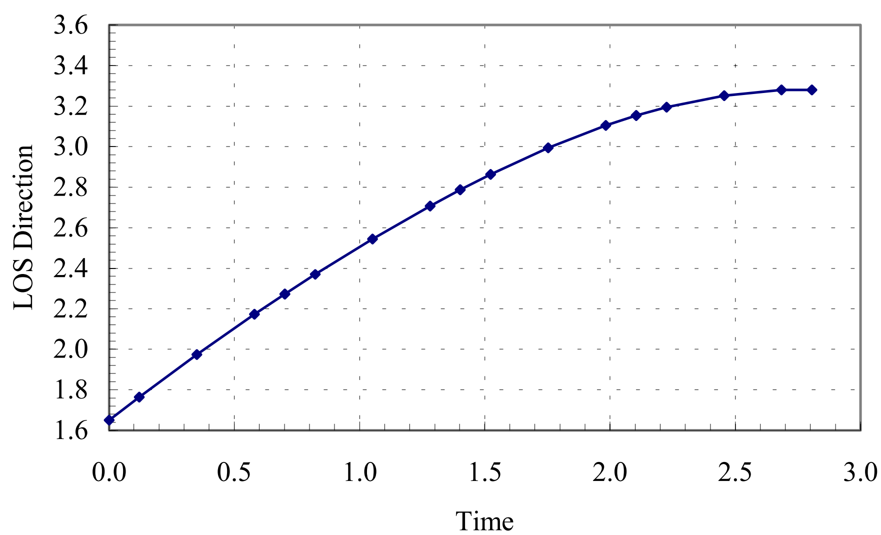

Figure 15. Line of Sight

Figs 13 to 15 report the result obtained with the DMM approach using normalized time and a normalized value for μ=1, the initial state vector for the target is [1.05,0.4,09.7590d-001] while the initial state vector for the pursuer is [1,0,0,1]]. In Fig.14 the thrust control law of both spacecrafts has been depicted along with optimality conditions (71) computed with the estimate of the adjoint variables provided by the NLP solver. Furthermore, it should be noted, as shown in

Fig.15, how the absolute direction of the line-of-sight from the pursuer to the target respects the optimal condition suggested by Guelman, Shinar and Green, i.e. both players turn toward the final line-of-sight direction asymptotically[2]. Both results confirm that the result obtained with DMM approach is optimal and equivalent to what can be computed with optimal control theory.

FINAL REMARKS

In this paper a novel approach to the design of optimal trajectory involving multiple objective functions and multiple players has been presented. The approach is fully direct and can solve efficiently formation flying problems and zero-sum differential games without resorting to differential algebraic equations coming from the optimal control theory. Here instead direct transcription is applied using multiple objective optimization theory and a multiphase approach to handle multiple players. The multiphase philosophy leads naturally to parallelization of the code in particular for mini-max problems with non-collaborative player for which a trade-off method is required. In this respect it has been shown, as the weighted sum method is equivalent to optimal control method when the players are collaborative while the trade-off method is equivalent when the players are not collaborative.

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