APPLICATION OF A DIRAC DELTA DIS-INTEGRATION TECHNIQUE TO THE STATISTICS OF ORBITING OBJECTS

Dario Izzo

Advanced Concepts Team, ESTEC, 2200 AG Noordwijk, The Netherlands

ABSTRACT

In many problems related to the description of the debris environment it is mandatory to be able to deal with orbital dynamics from a statistical point of view. This allows one to compute debris spatial densities, to evaluate collision probabilities and to improve the processing of the data received from observational campaigns. In this paper we use a novel technique based on the mathematical manipulation of Dirac's delta functions and we show how this is a powerful new tool in dealing with orbital mechanics from a statistical point of view. The methodology is applied to derive some analytical expressions for the density functions associated with the velocity distribution of orbiting objects.

Key words: Statistical Space Flight Mechanics; Diffused Satellite; Satellites Velocity Distribution; Uncertainties on Orbital Parameters.

1. INTRODUCTION

In this paper a novel methodology will be used to derive some expressions describing the probability density functions associated with the velocity distribution of orbiting objects. The objects are described by the statistical properties of their orbital parameters through probability distribution functions. Hypotheses of uncorrelated variables will be used but is not necessary. The work complements the already published results [Izzo (2005)] on the effect of orbital parameter uncertainties with new expressions on the velocities and some theoretical clarifications.

2. METHODOLOGY

To show the motivation and the benefits of the novel methodology used throughout this paper, we start, following Au & Tam (1999), by considering a discrete random variable Z with a probability distribution P given by the following:

Z	-3	$-\frac{1}{2}$	0	3	5	10
P	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{24}$	$\frac{1}{24}$

We then introduce a one-to-one transformation $Y' = Z + 1$, and a double-valued transformation $Y'' = Z^2$. It is immediately realized that the probability distribution for the new variables can be obtained by the following table:

$Y' = Z + 1$	-2	$\frac{1}{2}$	1	4	6	11
$Y'' = Z^2$	9	$\frac{1}{4}$	0	9	25	100
P	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{24}$	$\frac{1}{24}$

only in the case of the one-to-one transformation. For the other case some rearrangement is needed as $P(Y'' = 9) = P(X = -3) + P(X = 3)$.

The same problem exists for continuous random variables. For this reason the standard change of variable technique (for probability density functions) involves only monotonic transformations and a number of new variables equal to the number of old variables. These two fundamental shortcomings of the standard procedure may be solved by the introduction of a novel technique introduced independently, and in different contexts, by Au & Tam (1999) and Izzo (2002). The technique makes use of some fundamental properties of Dirac's delta functions, and in particular of the following theorem:

$$\delta[x - \hat{x}(y)] = \sum_i \frac{\delta[y - y_i]}{|\frac{d\hat{x}}{dy}|_{y=y_i}} \tag{1}$$

where y_i are all the solutions in y of the equation $x = \hat{x}(y)$. The theorem has also a multivariate form:

$$\delta[\mathbf{x} - \hat{\mathbf{x}}(\mathbf{y})] = \sum_i \frac{\delta[\mathbf{y} - \mathbf{y}_i]}{|J|_{\mathbf{y}=\mathbf{y}_i}} \tag{2}$$

where $J = \frac{\partial \hat{x}_i}{\partial y_j}$ is the Jacobian of the transformation $\hat{\mathbf{x}}(\mathbf{y})$ and $\mathbf{y}_i$ are the solutions to the equation $\mathbf{x} = \hat{\mathbf{x}}(\mathbf{y})$.

In the methodology used here Dirac's delta functions are used to define an initial probability density function associated with a deterministic process. The desired variables are then dis-integrated[1] from the process and the integral is easily evaluated by exploiting the quoted property of Dirac's delta functions.

[1] The term dis-integration refers here to a standard procedure in the theory of probability that is also called randomisation.

Let us consider for example the simple linear system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$. It is well known that its solution may be defined, by introducing the matrix exponential, as $\hat{\mathbf{x}}(\mathbf{x}_0, t) = e^{\mathbf{A}t}\mathbf{x}_0$. In this deterministic situation, where all the information needed to predict the evolution of the system is known, the probability density function associated with the state $\mathbf{x}$ may be written:

$$\rho(\mathbf{x}|\mathbf{x}_0, t) = \delta[\mathbf{x} - \hat{\mathbf{x}}(\mathbf{x}_0, t)]$$

We may now think of the initial condition $\mathbf{x}_0$ as a random variable with probability distribution function $f(\mathbf{x}_0)$ and we may therefore dis-integrate it by writing:

$$\rho(\mathbf{x}|t) = \int_{\mathbf{x}_0 \in D} \delta[\mathbf{x} - \hat{\mathbf{x}}(\mathbf{x}_0, t)] f(\mathbf{x}_0) d\mathbf{x}_0$$

where D is the domain of f. By making use of the multivariate version of the theorem on Dirac's Delta variable change stated by Eq.(2), it is possible to write:

$$\rho(\mathbf{x}|t) = \int_{\mathbf{x}_0 \in D} \delta[\mathbf{x}_0 - \mathbf{x}_0^*] f(\mathbf{x}_0^*) \frac{1}{|e^{\mathbf{A}t}|} d\mathbf{x}_0$$

where $\mathbf{x}_0^*$ is the only solution of equation $\mathbf{x} = \hat{\mathbf{x}}(\mathbf{x}_0, t)$ given by $\mathbf{x}_0^* = e^{At^{-1}}\mathbf{x}$. We are therefore able to write a final expression for the probability density function associated with the variable $\mathbf{x}$ whenever the initial conditions of the dynamical system that describe the time evolution of $\mathbf{x}$ are considered random and have to be dis-integrated from the process:

$$\rho(\mathbf{x}|t) = \frac{f(e^{At^{-1}})\mathbf{x}}{|e^{\mathbf{A}t}|}$$

Note that in this simple case the same expression could be reached by using a more standard technique of change of variable. The transformation $\mathbf{x} = e^{At}\mathbf{x}_0$ from $\mathbf{x}_0$ to $\mathbf{x}$ is in fact a one-to-one transformation (this is always the case for dynamical systems for which the Cauchy theorem holds) and the new variables $\mathbf{x}$ are as many as the old ones $\mathbf{x}_0$. The methodology outlined does not suffer from these limitations and can therefore be applied to general cases, in particular to cases in which we wish to consider time as random, and therefore also the most simple generic linear system considered above does not admit a one-to-one transformation.

3. APPLICATIONS TO SPACE FLIGHT MECHANICS

The methodology described in the previous section has already been applied to different problems related to orbital mechanics in general, and to the description of the space debris environment in particular, by Izzo (2002); Izzo & Valente (2004); Izzo (2005). The spatial density of a large family of orbiting objects has been derived analytically for Geostationary, LEO and Molnyia satellites, extending known results coming from past approaches [Izzo (2005)]. More results are derived here to show the use of Dirac's delta when dis-integrating random variables from a given stochastical process, and in particular in dealing with uncertainties related to the orbital parameters and their effects on velocity distributions.

3.1. The hodograph plane

In this section we briefly recall standard results on the velocity vector along a Keplerian orbit. In particular we start from the definitions of two important motion invariants:

$$\begin{array}{l} \vec{h} = \vec{r} \times \vec{v} \\ \mu\vec{e} = \vec{v} \times \vec{h} - \mu\frac{\vec{r}}{r} \end{array} \quad (3)$$

that is the angular momentum vector and the Laplace vector. By taking the vector product between these two quantities we easily get:

$$\frac{h}{\mu}\vec{v} = e \sin\nu \hat{i}_\rho + (1 + e\cos\nu)\hat{i}_\theta$$

where ν is the true anomaly along the orbit. The above expression describes the relation between the velocity and the true anomaly along a Keplerian orbit that will be used in the following.

For an orbit immersed in three dimensional space it is common to introduce the angle $\theta = \nu + \omega$ that will later be used in our calculations.

3.2. Velocity magnitude distribution

Let us consider a satellite on a Keplerian orbit and the probability density function associated with the variable $k = v^2$. Following the methodology outlined above, first a deterministic process is considered:

$$\rho(k|\vec{\alpha}, t) = \delta_k[k - \hat{k}(\vec{\alpha}, t)]$$

where $\vec{\alpha}$ is a vector containing the initial conditions defining the Keplerian orbit (any set of orbital parameters might be considered) and $\hat{k}(\vec{\alpha}, t) = \mu\left\{\frac{2}{p}[1 + e\cos[\theta(t) - \omega]] - \frac{1}{a}\right\}$. Then the time t is considered as random and uniformly distributed over one orbital period so that the variable is dis-integrated from the process:

$$\rho(k|\vec{\alpha}) = \frac{1}{T}\int_0^T \delta_k[k - \hat{k}(\vec{\alpha}, t)]dt$$

To evaluate the above integral we must change the variable in the Dirac delta function and we do this by applying the fundamental property of Dirac's delta functions stated in Eq.(1):

$$\rho(k|\vec{\alpha}) = \frac{1}{T}\int_0^T \sum_i \frac{\delta_t[t-t_i]}{\left|\frac{d\hat{k}}{dt}\right|_{t=t_i}} dt \qquad (4)$$

where we must sum all the solutions in t to the equation $\hat{k}(\vec{\alpha}, t) = k$. Simple astrodynamics tell us that these solutions are given by

$$\cos[\theta(t_i) - \omega] = \frac{2k - v_a^2 - v_p^2}{v_p^2 - v_a^2}$$

where v_a and v_p are the apogee and perigee velocities related to the semi-major axis and eccentricity by the relations:

$$e = \frac{v_p - v_a}{v_p + v_a}, \quad a = \frac{\mu}{v_a v_p}$$

Two of the above solutions will be contained within the integration interval of one period, giving non-zero terms in equation 4. Corresponding to these two solutions we also have:

$$\sin[\theta(t_i) - \omega] = \pm\frac{2}{v_p^2 - v_a^2}\sqrt{(k - v_a^2)(v_p^2 - k)}$$

and

$$\frac{1}{r} = \frac{1}{2\mu}(k + v_a v_p)$$

so that we may evaluate the derivatives:

$$\begin{aligned}\left|\frac{d\hat{k}}{dt}\right|_{t=t_i} =& \frac{2\mu}{p} e \sin[\theta(t_i) - \omega]\dot{\theta}(t_i) \\ =& \frac{2\mu}{p} e \sin[\theta(t_i) - \omega] b \sqrt{\frac{\mu}{a}}\frac{1}{r^2} \\ =& ... \\ =& \frac{(k + v_a^2 v_p^2)^2}{2\mu(v_a + v_p)}\sqrt{(k - v_a^2)(v_p^2 - k)} \qquad (5)\end{aligned}$$

Plugging this expression into eq.(4) we get the final expression:

$$\rho(k|v_a, v_p) = \frac{2}{\pi}\frac{v_p + v_a}{(k + v_a v_p)^2}\sqrt{\frac{v_p^3 v_a^3}{(k - v_a^2)(v_p^2 - k)}}$$

This is plotted in Figure 1 for an Earth orbit.

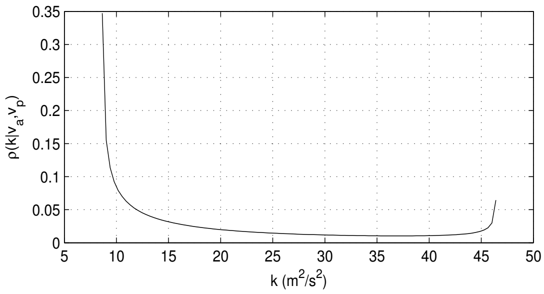

Figure 1. Probability density function associated with the variable $k = v^2$ on an Earth orbit having $e = 0.4$ and $a = 20000km$.

3.3. Radial velocity distribution

We show here the calculation that leads to determining the analytical expression for the probability density function associated with the variable v_ρ, i.e. the radial velocity along an orbit, if the time is considered as random and uniformly distributed within an orbital period. We start from a deterministic process by writing:

$$\rho(v_\rho|\vec{\alpha}, t) = \delta_v[v_\rho - \hat{v}_\rho(\vec{\alpha}, t)]$$

where $\hat{v}_\rho(\vec{\alpha}, t) = \frac{\mu e}{h}\sin[\theta(t) - \omega]$. We then consider time as random with a uniform probability distribution function, and we dis-integrate it from the process by writing:

$$\rho(v_\rho|\vec{\alpha}) = \frac{1}{T}\int_0^T \delta_v[v_\rho - \hat{v}_\rho(\vec{\alpha}, t)]dt$$

We then make use of the theorem on Dirac's delta functions stated by Eq.(1):

$$\rho(v_\rho|\vec{\alpha}) = \frac{1}{T}\int_0^T \sum_i \frac{\delta_t[t - t_i]}{\left|\frac{d\hat{v}_\rho}{dt}\right|_{t=t_i}} dt$$

where t_i are the solutions in t to the equation $\hat{v}_\rho(\vec{\alpha}, t) = v_\rho$ given by the expression:

$$\sin[\theta(t_i) - \omega] = \frac{v_\rho}{v_{\rho_M}}$$

where we introduced the maximum value of the radial velocity $v_{\rho_M} = \frac{\mu e}{h}$. We also have:

$$\begin{aligned}\left|\frac{d\hat{v}_\rho}{dt}\right|_{t=t_i} =& v_{\rho_M}\cos[\theta(t_i) - \omega]\dot{\theta} \\ =& v_{\rho_M}\cos[\theta(t_i) - \omega] b \sqrt{\frac{\mu}{a}}\frac{1}{r^2} \\ =& ... \\ =& \frac{2\pi}{T}\frac{\sqrt{v_{\rho_M}^2 - v_\rho^2}}{\sqrt{(1 - e^2)^3}}\left(1 \pm \frac{e}{v_{\rho_M}}\sqrt{v_{\rho_M}^2 - v_\rho^2}\right)^2 \qquad (6)\end{aligned}$$

Note that the values of the derivative are different for the two different time instants t_i contained in one orbital period. After some more manipulations we get:

$$\rho(v_\rho|v_{\rho_M},e)=\frac{1}{\pi}\frac{v_{\rho_M}^2\sqrt{(1-e^2)^3}}{\sqrt{v_{\rho_M}^2-v_\rho^2}}\frac{v_{\rho_M}^2+e^2(v_{\rho_M}^2-v_\rho^2)}{\left(v_{\rho_M}^2-e^2(v_{\rho_M}^2-v_\rho^2)\right)^2}$$

This is plotted in Figure 2 for an Earth orbit.

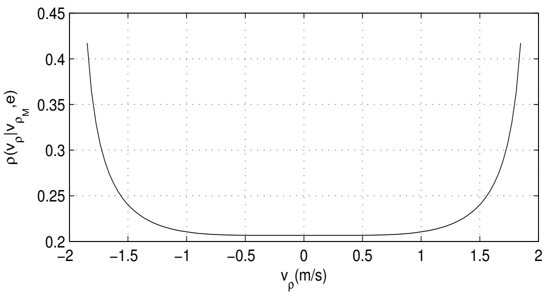

Figure 2. Probability density function associated with the radial velocity on an Earth orbit having $e = 0.4$ and $a = 20000km$.

3.4. Radial-Tangential velocity distribution

Having acquired confidence with the new methodology and with its application to space flight mechanics problems, we are now ready to build more complicated probability distribution functions. In problems related to collisions between objects belonging to different families of orbits, it is important to be able to describe somehow the relative velocity between objects at the time of possible impact. Many other problems require to be able to write the probability density function associated with an object velocity. We here derive a first result on this issue. In a deterministic process the probability density function associated with the variables v_ρ and v_θ may be written as:

$$\rho(v_\rho,v_\theta|\vec{\alpha},t)=\delta[v_\rho-\hat{v}_\rho(\vec{\alpha},t)]\delta[v_\theta-\hat{v}_\theta(\vec{\alpha},t)]$$

where

$$\hat{v}_\rho=v_{\rho_M}\sin[\theta(t)-\omega]$$
$$\hat{v}_\theta=v_{\rho_M}\left\{\tfrac{1}{e}+\cos[\theta(t)-\omega]\right\}$$

We now consider the time as a random variable uniformly distributed in one orbital period, and the eccentricity as a random variable with which we associate a probability distribution function $f(e)$. By dis-integrating these two variables we get:

$$\rho(v_\rho,v_\theta|v_{\rho_M})=\frac{1}{T}\int_0^1\int_0^T\rho(v_\rho,v_\theta|\vec{\alpha},t)f(e)dtde$$

We could now use the multivariate form of the Dirac delta property on variable change stated by eq.(2). This way, though, we would be forced to evaluate the determinant of the Jacobian matrix. By applying eq.(1) two times in cascade we actually save some calculations. We start therefore by eliminating the Dirac delta in v_ρ. Taking advantage of the calculations already done in the previous paragraph we get:

$$\rho(v_\rho,v_\theta|v_{\rho_M})=$$

$$=\int_0^1\frac{1}{2\pi}\frac{\sqrt{(1-e^2)^3}}{\sqrt{v_{\rho_M}^2-v_\rho^2}}\frac{f(e)\delta[v_\theta-\hat{v}_{\theta_1}]}{\left(1+\frac{e}{v_{\rho_M}}\sqrt{v_{\rho_M}^2-v_\rho^2}\right)^2}+$$

$$+\frac{1}{2\pi}\frac{\sqrt{(1-e^2)^3}}{\sqrt{v_{\rho_M}^2-v_\rho^2}}\frac{f(e)\delta[v_\theta-\hat{v}_{\theta_2}]}{\left(1-\frac{e}{v_{\rho_M}}\sqrt{v_{\rho_M}^2-v_\rho^2}\right)^2}de$$

where $\hat{v}_{\theta_{1,2}}(v_{\rho_M},e)=\frac{v_{\rho_M}}{e}\pm\sqrt{v_{\rho_M}^2-v_\rho^2}$. We now apply eq.(1) again to eliminate the two Dirac deltas in v_θ. By taking into account that:

$$\left|\frac{d\hat{v}_{\theta_{1,2}}}{de}\right|=\frac{v_{\rho_M}}{e^2}\tag{7}$$

and that the two solutions in e to the two equations $v_\theta=\hat{v}_{\theta_{1,2}}$ are:

$$e_{1,2}^*=\frac{v_\theta}{v_{\rho_M}\mp\sqrt{v_{\rho_M}^2-v_\rho^2}}$$

we get a final expression of the form:

$$\rho(v_\rho,v_\theta|v_{\rho_M})=$$

$$=\frac{1}{2\pi}\frac{v_{\rho_M}}{v_\theta^2\sqrt{v_{\rho_M}^2-v_\rho^2}}\sum_{i=1}^2\sqrt{(1-e_i^{*2})^3}f(e_i^*)\tag{8}$$

The expression above still contains v_{ρ_M} as a conditioning variable. This variable might also be dis-integrated leading to a formula solvable by quadrature.

3.5. One last example

As a last example we here derive a quite remarkable formula for the probability distribution function associated with the variables r,v_ρ,v_θ.

$$\rho(r,v_\rho,v_\theta|\vec{\alpha},t)=$$
$$\delta[r-\hat{r}(\vec{\alpha},t)]\delta[v_\rho-\hat{v}_\rho(\vec{\alpha},t)]\delta[v_\theta-\hat{v}_\theta(\vec{\alpha},t)]$$

Following the usual methodology, we first consider the time as random and uniformly distributed in one period, and then the eccentricity as random and described by a probability density function f. We get the expression:

$$\rho(r, v_\rho, v_\theta | v_{\rho_M}) = \left(\frac{1}{2\pi} \frac{v_{\rho_M}}{v_\theta^2 \sqrt{v_{\rho_M}^2 - v_\rho^2}} \right) * \left(\sum_{i=1}^{2} \sqrt{(1 - e_i^{*2})^3} f(e_i^*) \delta[r - \hat{r}_i(v_\theta, v_{\rho_M})] \right) \quad (9)$$

where $\hat{r}_i(v_\theta, v_{\rho_M}) = \frac{\mu e_i^*}{v_{\rho_M}} \frac{1}{v_\theta}$. We now take one last step by considering v_{ρ_M} as random and with a probability distribution function g and we eliminate the last Dirac delta by dis-integrating the random variable from the orbital process. By taking into account that:

$$\left| \frac{d\hat{r}_i}{dv_{\rho_M}} \right| = \frac{\mu}{v_{\rho_M}^2 \sqrt{v_{\rho_M}^2 - v_\rho^2}}$$

and that the only solution in v_{ρ_M} of the equation $r = \hat{r}_i$ is:

$$v_{\rho_M}^* = \frac{\mu}{\sqrt{2\mu r - r^2 v_\rho^2}}$$

we get, applying again the Dirac delta property stated in eq.(1) and after some basic algebraic manipulations, a final expression in the form:

$$\rho(r, v_\rho, v_\theta) = \\ = \frac{\mu^2}{2\pi} \frac{1}{v_\theta^2} \frac{g(v_{\rho_M}^*)}{\sqrt{(2\mu r - r^2 v_\rho^2)^3}} \sum_{i=1}^{2} \sqrt{(1 - e_i^{*2})^3} f(e_i^*) \quad (10)$$

This probability density function is no longer conditioned by the knowledge of any initial condition. We have managed to describe the velocity distribution at a certain distance of a satellite, or a group of objects, whose uncertain orbits are described via the density functions associated with two orbital parameters, the eccentricity e and the maximum radial velocity ρ_M. The above equation is valid under the hypothesis that these two random variables are uncorrelated, but a similar expression may also be obtained if a joint probability distribution function is available.

4. CONCLUSIONS

The technique based on the dis-integration of the orbital parameters from an initial density function written in terms of Dirac's delta proves to be very useful for deriving the velocity distributions of a family of objects. Some new formulas have been derived here that the author hopes will prove to be useful tools improving our understanding of the debris environment and of all those situations in which orbiting objects have to be described by a probabilistic distribution rather than by a certain position and velocity.

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