A Second-Order Gradient Solver using a Homotopy Method for Space Trajectory Problems

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Recent developments in low-thrust trajectory optimization methods have shown that second-order gradient techniques can be efficient. Robust second-order methods allow one to solve complex dynamical problems, accounting for perturbations, and path constraints, with high fidelity. Convergence of the method, however, requires a very good initial guess, or a lot of user experience. Based on a quadratic expansion of the Lagrangian for the optimal control problem, the proposed method uses well known homotopy and continuation techniques to simplify the need of a very good initial guess for the optimization problem. The continuation variable is assigned to a physical parameter to solve complex dynamical problems. The new algorithm demonstrate an increased robustness.

Nomenclature

$\boldsymbol{\lambda}$	Co-State vector
$\boldsymbol{\nu}$	Lagrange vector
$\boldsymbol{\psi}$	Terminal constraints
$\delta\mathbf{u}$	Control update
Δs	arc length, step change
$\mathbf{f}$	Dynamics
$\mathbf{H}$	Homotopy map
$\mathbf{K}$	Sensitivity matrix
$\mathbf{Q}$	Sensitivity matrix
$\mathbf{R}$	Ricatti matrix
$\mathbf{T}$	Sensitivity matrix
$\mathbf{u}$	Control
$\mathbf{V}$	Sensitivity matrix
$\mathbf{x}$	State
τ	Homotopy variable
H	Hamiltonian
n_k	Dimension of the constraint vector
n_p	Dimension of the parameter vector
n_u	Dimension of the control vector
n_x	Dimension of the state vector
s	arc length or curvilinear abscissa

I. Introduction

Trajectory optimization and optimal control theory applied to low-thrust space trajectory problems have always nurtured many different algorithms depending on the characteristics of the dynamics. Betts[1] presents an overview of most of the methods which have been applied to space trajectory optimization, both direct and indirect methods. Direct methods[2, 1] can be characterized as non-linear programming methods, where

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the state and the control are discretized in the time and considered as the decision variables. This is achieved at the cost of reaching only mid-fidelity dynamical satisfaction but a high radius of convergence. Indirect methods[3], on the other hand, transform the optimal control problem into a boundary value problem where the dynamics are integrated precisely, but they usually suffer from a small radius of convergence.

Second order methods have been used for space trajectory optimization. Although, more difficult to implement, they are generally more robust, with a disposition to take the advantages of both direct and indirect methods. As defined in most recent work [4, 5], they both satisfy the dynamics with high fidelity and provide good convergence properties, even though they suffer from slow convergence[6].

Many different second order algorithms have appeared over recent years, and are mainly based on differential dynamic programming[7] (DDP). For instance, Colombo [8] used the DDP algorithm and applied it to the design of trajectories to asteroids, simplifying the control with a quadratic cost to prevent any bang-bang structure and to improve the convergence. Lantoine [9] proposed a simplification of the dynamical model, allowing the use of an analytical propagation, improving the speed and convergence of the algorithm. Whiffen[4, 10] proposed an improvement of the DDP algorithm with static parameters and provided a generic tool for high-fidelity trajectory optimization. The tool itself has been used in the design of currently flying missions.[11].

In Ref.[12], the author proposed a different formulation of the second order algorithm based on Refs. [6, 13], and applies it to the difficult multi-phase problem of a spacecraft going from a planet-centered motion to another planet-centered motion (escape and capture spirals with interplanetary phase) without formulating specific intermediate conditions and constraints. This is done without using patching trajectory segment methods, thus transitioning between different dynamical regimes with high fidelity. The algorithm thus proves to be robust and able to cope with complex dynamics.

More generally, second order methods solve dynamical problems with control and state constraints. Thus we should expect a greater ability to exploit dynamical interactions, such as swing-bys in space trajectory problems for instance. However, convergence still requires a good initial guess, or a lot of user experience, for hard dynamical problems. Some work has thus been done on providing good initial guesses for complex problems for use in second-order methods[14]. The paper presents an approach that uses the primer vector as a baseline to provide the optimal control.

This study is about proposing an approach that would be able to solve trajectory problems in complex dynamical environments (e.g. three body problems, interplanetary transfers with swing-bys, without the explicit formulation of the intermediate constraints), and with poor initial guess requirements. The principal attempt of this study is to use continuation and homotopy methods[15, 16] with the second order algorithm[12] to solve optimal control problems and reduce the need of good initial guesses.

II. Homotopy Method

II.A. Homotopy Approach

For the sake of completeness, basics of homotopy methods are recalled, and a more comprehensive review can be found in Ref. [15]. Many codes are available (HOMPACK[16], PITCON[17], ALCON).

Homotopy methods usually apply to the root finding problem. In optimal control, such root finding problem appears when formulating a two point boundary value problem, using for instance an indirect method approach. The desired root finding problem $\mathbf{F}(\mathbf{x}) = 0$ is transformed into another, simpler, root finding problem $\mathbf{G}(\mathbf{x}) = 0$ for which a solution is known. The transformation is parameterized with a homotopy parameter, denoted here as τ, and a homotopy map $\mathbf{H}(\mathbf{x}, \tau)$ is constructed. For instance, the following convex homotopy map $\mathbf{H}(\mathbf{x}, \tau) : \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_k}$ can be posed[15]

$$\mathbf{H}(\mathbf{x}^0, \tau = 0) = \mathbf{G}(\mathbf{x}^0) \tag{1}$$

$$\mathbf{H}(\mathbf{x}^*, \tau = 1) = \mathbf{F}(\mathbf{x}^*) \tag{2}$$

$$\mathbf{H}(\mathbf{x}, \tau) = (1 - \tau)\mathbf{G}(\mathbf{x}) + \tau\mathbf{F}(\mathbf{x}) \tag{3}$$

where $\mathbf{x}_0$, and $\mathbf{x}^*$, are respectively the solutions of the simplified problem $\mathbf{G}$ and the nominal problem $\mathbf{F}$. Note then, that $(\mathbf{x}^0, \tau = 0)$, and $(\mathbf{x}^*, \tau = 1)$ are both solutions of the homotopy function $\mathbf{H}$, and belong to the curve, or manifold, $\mathbf{H}^{-1}(0)$. This curve can then be followed from $\mathbf{x}_0$ to find the desired solution

$\mathbf{x}^*$. The homotopy parameter τ is varied and for each iterate τ_i, an instance of the problem, $\mathbf{H}(\mathbf{x}, \tau_i) = 0$, is solved using the former iterate problem solution as initial guess. As a result, the user provides only an initial guess to the simple problem $\mathbf{G}$, which is eventually modified by the homotopy method to match the original complex problem $\mathbf{F}$. The variations in the homotopy parameter can follow a specific homotopy path to obtain global convergence.

II.B. Homotopy Continuation Methods

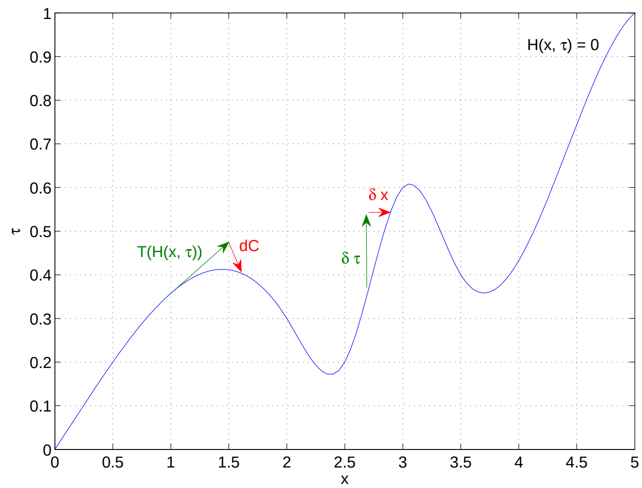

Figure 1. Illustration of the path following method

When the homotopy parameter τ is varied discretely and increasingly, with a constant step, the method is called continuation, while homotopy refers to the case when τ is updated according to the curvature of the path $\mathbf{H}^{-1}(0)$. Then, path tracking techniques allow the following of a solution path $\{\mathbf{x}, \tau\}_i$, going from $\mathbf{x}^0$ to $\mathbf{x}^*$. Homotopy methods rely on the path-following techniques used. We can mention for instance: differential continuation and simplicial homotopy methods.

Differential continuation methods use information on the Jacobian of the homotopy map with respect to the homotopy parameter for the prediction step $(\delta\mathbf{x}, \delta\tau)$, and they are based on the Newton's method (Predictor-corrector) which provides correction steps dC to get back on the curve, as illustrated on Fig. 1. One major assumption is that the zero path is described by a differential curve, thus providing some regularity properties on the curve to track. The advantage of this method is usually quick convergence owing to the good step updates.

Simplicial methods use an approximation of the zero path as a piecewise-linear curve. A simplex is constructed to find the root of each iterate problem. This method is more robust than differential continuation, but it is also slower. No particular conditions are required on the regularity of the zero path.

The current study focuses on the differential continuation approach.

II.C. Differential Homotopy Continuation Method

II.C.1. Path Tracking

Because the zero curve is not necessarily monotonous, or can have turning points (as pictured in Fig. 1), it is actually convenient to work with the arc length s instead of τ directly. We thus write $\mathbf{H}(\mathbf{x}(s), \tau(s))$ for curve tracking, differentiate with respect to the arc length s, and get the tangent vector that gives the direction to follow.

The curve $(\mathbf{x}(s), \tau(s)) \subset \mathbf{H}^{-1}(0)$ is the solution of the equivalent initial value problem[18]:

$$\frac{d\mathbf{H}}{ds} = T\left(\frac{d\mathbf{H}}{d\mathbf{x}}(\mathbf{x}(s))\right) \tag{4}$$

$$\mathbf{x}(0) \in \mathbf{H}^{-1}(0) \tag{5}$$

where $T(A)$ would denote the tangent vector to an mapping A. This is illustrated on Fig. 1. Basically, any ordinary differential equation solver can tackle the problem of Eq. (4). More conveniently however, the tangent vector is given by

$$\frac{d\mathbf{H}}{ds} = \frac{d\mathbf{H}}{d[\mathbf{x}, \tau]}\frac{d[\mathbf{x}, \tau]}{ds} = 0 \tag{6}$$

which yields

$$\begin{bmatrix} \frac{d\mathbf{x}}{ds} \\ \frac{d\tau}{ds} \end{bmatrix} \in \ker\left(\frac{d\mathbf{H}}{d[\mathbf{x}, \tau]}\right) \tag{7}$$

and the tangent can be chosen such that,

$$\left\| \begin{bmatrix} \frac{d\mathbf{x}}{ds} \\ \frac{d\tau}{ds} \end{bmatrix} \right\| = 1 \tag{8}$$

with the derivatives $\frac{d\mathbf{H}}{d[\mathbf{x},\tau]}$ obtained analytically or numerically. The tangent vector and the update on τ are then computed for a given step increase Δs. In that way, the homotopy parameter does not necessarily increase to ensure a better curve tracking.

Because the step length Δs cannot be computed efficiently, the next initial guess point is unlikely to stay on the tracked curve, and thus correction steps may be required to go back on the curve.

II.C.2. Necessary Conditions

There are necessary conditions for a homotopy path to exist:

1. the Jacobian matrix $\mathbf{H}_x(x, \tau)$ has rank n, where n is the dimension of $\mathbf{x}$, on the manifold $\mathbf{H}^{-1}(0)$. This condition ensures that the path does not cross itself.
2. $\mathbf{H}(\mathbf{x}, \tau = 0)$ has a unique solution.
3. $\mathbf{H}$ is bounded, or smooth.

Often, homotopy continuation methods are termed probability-one homotopy methods because if a zero-path exists (almost always) then following the path always leads to a solution of the problem.

For this study, we assume that all these conditions are satisfied. They can be ensured easily in part considering the space trajectory problem dynamics, constraints and objective function are continuous, and by formulating consistent boundary constraints.

II.D. Continuation with Respect to Physical Parameters

Continuation and homotopy are often used when a solution to a problem is known, and the search for an initial guess of a more complex problem is tedious. In this case, the initial conditions or problem parameters are changed. For instance, in the case of low-thrust trajectory problems, the homotopy parameter can be the thrust amplitude[19], a gravitational parameter[20, 21], dates[22], etc... A similar approach applied to space trajectory problem can be found in Ref. [23]. Often the homotopy consists of changing a finite set of variables. In Ref. [22] however, the authors use directly a solution trajectory as initial guess, instead of the usual costate vector. This approach seems particularly well suited for discrete time methods (direct methods, collocation, multiple shooting, ...). Furthermore, a continuation scheme is proposed for both the terminal constraints and the objective value function.

When the homotopy variable is a dynamical parameter the method can basically allow a different exploration of interesting domains of the solution space, provide more controllability, and eventually return a-priori non-trivial solutions. For instance, in a interplanetary space trajectory problem an advantage of continuation with respect to the gravitational parameter is the a priori weak initial guess requirement[23], and the conservation of the angular range[20]. Solutions to the zero-gravity case are readily found. The conservation of angular range fixes the problem geometry, which is particularly suitable for space interplanetary trajectory problems where phasing is an issue (e.g. for rendezvous, swingbys and flybys). However, a deformation of the dynamical problem can bring other theoretical issues that prevent easily finding any solutions.

It is known that second order gradient methods can solve difficult optimal control problems (in the sense of complex dynamics)[12, 4]. Combining both approaches, the gradient method with a physical parameter homotopy technique, one may be able to actually find interesting solutions, such as optimal swing-by maneuvers, low-energy transfers, ... The purpose of the present study is thus to propose a method that would simplify the initial guess need for difficult optimal control problems.

III. Modified Gradient Method with Homotopy Continuation

III.A. Problem Description

Consider a general continuous space trajectory problem. The dynamics are denoted:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u}, p; t) \tag{9}$$

with state $\mathbf{x}$ and, control $\mathbf{u}$ such that

$$\|\mathbf{u}\| \leq 1 \tag{10}$$

and a parameter p.

A set of initial conditions is given by:

$$\phi(\mathbf{x}(t_0), p) = 0 \tag{11}$$

And terminal constraints of the form:

$$\psi(\mathbf{x}(t_f)) = 0 \tag{12}$$

The optimization is about the minimization of an objective function written in Mayer form:

$$\min_{\mathbf{u}} J(\mathbf{u}, p) \tag{13}$$

The objective function is minimized with respect to the control $\mathbf{u}$. The parameter p is present to introduce a path following method and will play the role of the homotopy parameter. Consequently, p does not need to minimize J if the homotopy map describes only the constraint satisfaction and not the first order optimality conditions.

III.B. Sensitivity Equations and Control Update

The following developments can be found in the work of the author, and Refs.[12, 5]. They are briefly recalled here for completeness. For this problem, consider the extended value function:

$$V(\mathbf{u}, \boldsymbol{\nu}, \tau) = J(\mathbf{u}, \tau) + \boldsymbol{\eta}^T \boldsymbol{\phi}(\mathbf{x}(t_0), \tau) + \boldsymbol{\nu}^T \boldsymbol{\phi}(\mathbf{x}(t_f)) + \boldsymbol{\psi}(\mathbf{x}(t_f))^T C_p \boldsymbol{\psi}(\mathbf{x}(t_f)) \tag{14}$$

where $\boldsymbol{\nu}$ is the Lagrange vector for the constraints, and C_p is a regularization matrix.

The Hamiltonian is simply:

$$H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}, \tau; t) = \boldsymbol{\lambda}^T \mathbf{f}(\mathbf{x}, \mathbf{u}, \tau; t) \tag{15}$$

as the objective does not include Lagrangian terms (integral terms).

The following transformations are used

$$\delta\boldsymbol{\lambda} = \mathbf{R}\boldsymbol{\delta x} + \mathbf{K}d\boldsymbol{\nu} \tag{16}$$

$$d\boldsymbol{\psi} = \mathbf{K}^T\boldsymbol{\delta x} + \mathbf{Q}d\boldsymbol{\nu} \tag{17}$$

with $\mathbf{R}(t) \in M_{n_x,n_x}(\mathbb{R})$, $\mathbf{K}(t) \in M_{n_x,n_k}(\mathbb{R})$, and $\mathbf{Q}(t) \in M_{n_k,n_k}(\mathbb{R})$.

A feedback update of the control is constructed with the form

$$\delta \mathbf{u} = \boldsymbol{\alpha} + \boldsymbol{\beta} \delta \mathbf{x} + \boldsymbol{\omega} d\boldsymbol{\nu} \tag{18}$$

with:

$$\boldsymbol{\alpha} = -H_{uu}^{-1} H_u \tag{19}$$

$$\boldsymbol{\beta} = -H_{uu}^{-1} \left(H_{xu} + \mathbf{R}^T \mathbf{f}_u\right) \tag{20}$$

$$\boldsymbol{\omega} = -H_{uu}^{-1} \mathbf{K}^T \mathbf{f}_u \tag{21}$$

with $\boldsymbol{\alpha}(t) \in \mathbb{R}^{n_u}$, $\boldsymbol{\beta}(t) \in M_{n_u,n_x}(\mathbb{R})$, and $\boldsymbol{\omega}(t) \in M_{n_u,n_k}(\mathbb{R})$, and $\mathbf{R}$, $\mathbf{K}$, $\mathbf{Q}$ are provided by the set of differential equations:

$$\frac{d\boldsymbol{\lambda}^T}{dt} = -H_x + H_u H_{uu}^{-1} \left(H_{ux} + H_{u\lambda} \mathbf{A}\right) \tag{22}$$

And:

$$-\frac{d\mathbf{R}}{dt} = H_{xx} + \mathbf{R}^T \mathbf{f}_x + \mathbf{f}_x \mathbf{R} + \beta^T \left(H_{ux} + \mathbf{f}_u \mathbf{R}\right) \tag{23}$$

$$-\frac{d\mathbf{K}}{dt} = \mathbf{K}^T \mathbf{f}_x + \omega^T \left(H_{ux} + \mathbf{f}_u \mathbf{R}\right) \tag{24}$$

$$-\frac{d\mathbf{Q}}{dt} = \mathbf{K}^T \mathbf{f}_u \omega \tag{25}$$

The matrix $\mathbf{Q}$ allows to compute an update of the Lagrange multiplier $\boldsymbol{\nu}$. The quantities H_{uu}, H_u, H_{ux}, $H_{u\lambda}$ depend on $\boldsymbol{\lambda}$ and the derivatives of $\mathbf{f}$.

These ODEs must be integrated backward, using the terminal conditions:

$$\boldsymbol{\lambda}(t_f) = \frac{\partial J}{d\mathbf{x}_f} + \boldsymbol{\nu}^T \frac{\partial \psi}{\partial \mathbf{x}_f} + \frac{\partial \psi}{\partial \mathbf{x}_f}^T C_p \psi \tag{26}$$

And:

$$\mathbf{R}(t_f) = J_{xx} + \boldsymbol{\nu}^T \psi_{xx} + \psi^T C_p \psi_{xx} + \psi_x^T C_p \psi_x \tag{27}$$

$$\mathbf{K}(t_f) = \psi_x \tag{28}$$

$$\mathbf{Q}(t_f) = 0 \tag{29}$$

Once the ODE of the sensitivity matrices are integrated, an update on the control can be computed with Eq. (18). Then, the state dynamical equations are integrated forward with the updated control. If the second order developments are valid, this update leads to an improvement of the extended value function, a subsequent reduction of the objective function and eventually a better satisfaction of the constraints.

III.C. Gradient Method with Homotopy Parameter

Based on the gradient method, a homotopy parameter is included to account for the deformation of the problem. It is assumed that the algorithm works in steps such that the prediction and the correction back to the curve are done concurrently. With sufficiently small updates, the gradient algorithm is expected to follow the zero curve closely.

The method is based on the predictor-corrector technique of the differential homotopy continuation method. However, the second order algorithm does not involve any root finding problem in its current form, because the optimality conditions are not expressed as constraints. The optimality of the solution is not tackled by the homotopy method itself. Fortunately, the second order method, once provided a feasible solution can efficiently and rapidly reach the optimality conditions. Concentrating only on state constraints satisfaction still provides a valid root finding problem, suitable for the homotopy procedure.

Considering the problem variables $\{\mathbf{u}, \tau\}$ and its size, it would be inconvenient to compute any Jacobian to follow the zero-curve. Indeed, as a consequence of the Maximum Principle and Eqs. (16), (17), the maximizing control $\mathbf{u}(t)$ can be traded with the costate vector $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}(t_0)$. The working variables can thus be reduced to $\{\boldsymbol{\lambda}_0, \tau\}$ only. This results in a Jacobian of smaller dimension. This trade discards the second order part, but it can be taken care of with the correction step.

The parameter τ must evolve within $[0, 1]$, the initial value being 0. Based on the previous developments, a gradient-based homotopy method can be developed, noting,

$$\tilde{\boldsymbol{\psi}} = \mathbf{H}(\mathbf{x}_f, \tau) \tag{30}$$

Many different homotopy maps are possible, and the difficulty of homotopy continuation methods is often to find a map that is both simple to deform and provides the desired exploration of the search space.

The new constraint vector $\tilde{\boldsymbol{\psi}}$ defines the terminal state constraints for the modified gradient method. But to apply the homotopy technique, the mapping used is actually applied to both the terminal state constraints and the terminal transversality conditions, even though the latter are implicitly satisfied by the gradient method (see Eq. (26)). This is necessary to compute a Jacobian that would be of full rank to construct a zero continuation path. Indeed, because of the tangent vector, the dimension of the null space is necessarily one. By the theorem of dimension, the dimension of the constraint space must be of the dimension of the decision vector space. This is the case for optimal control problems when considering the transversality conditions.

The update of the homotopy parameter τ thus follows an update of the curvilinear abscissa s, as Eq. (7). The update is computed using the sensitivities

$$\frac{d\boldsymbol{\psi}}{d\boldsymbol{\lambda}_0} = \mathbf{Q}_0 \left(\mathbf{K}_0^T \mathbf{K}_0\right)^{-1} \mathbf{K}_0^T \tag{31}$$

$$\frac{d\boldsymbol{\lambda}_f}{d\boldsymbol{\lambda}_0} = \mathbf{K}_f \left(\mathbf{K}_0^T \mathbf{K}_0\right)^{-1} \mathbf{K}_0^T \tag{32}$$

The derivative with respect to the homotopy variable may be more complicated to compute if the dynamics depends on τ. In this case, consider the additional ODEs,

$$-\frac{d\mathbf{V}}{dt} = \mathbf{K}^T \left(\mathbf{f}_u H_{uu}^{-1} (H_{pu} + \mathbf{T}\mathbf{f}_u)^T + \mathbf{f}_p\right) \tag{33}$$

$$-\frac{d\mathbf{T}}{dt} = H_{px} + \mathbf{T}^T \mathbf{f}_x + \mathbf{K}\mathbf{f}_u \left(H_{ux} + \mathbf{f}_u \mathbf{R}\right) + \mathbf{f}_p \mathbf{R} \tag{34}$$

with $\mathbf{V}(t_f) = 0$ and $\mathbf{T}(t_f) = 0$. This yields to the sensitivity $\nabla_\tau \boldsymbol{\psi} = \mathbf{V}(t_0)$. Considering the dimension and the sensibility of the problem, this derivative can also be computed by finite differences to avoid the sometimes expensive integration of $\mathbf{V}$ and $\mathbf{T}$.

A SVD can thus be used to compute the null space of $\nabla_{x,\tau}\mathbf{H}$. The tangent direction $T(\mathbf{H})$ of the path with respect to s is a base of this null space[24].

$$T(\mathbf{H}(\mathbf{x}, \tau(s))) = \begin{bmatrix} \frac{d\mathbf{x}}{ds} \\ \frac{d\tau}{ds} \end{bmatrix} \tag{35}$$

The orientation of the tangent vector is determined by considering the previous iterative tangent vector. The two should point in the same average direction, and thus the angle between the two should be smaller than $\pi/2$.

And, the update of the homotopy parameter τ is,

$$d\tau = \min\left\{\frac{d\tau}{ds}\Delta s, 1 - \tau^i\right\} \tag{36}$$

$$\tau^{i+1} = \tau^i + d\tau \tag{37}$$

where Δs is a fixed step. Small Δs ensures accurate curve tracking, but many iterations.

The iterative process is twofold: first, the method provides an update of the homotopy parameter and the control to move forward on the curve, then, the gradient method provides correction steps to get back on the curve. Thus, once a solution to the problem with the initial value of the homotopy parameter is found, the algorithm performs a curve tracking of the path $(\mathbf{u}(t,\tau),\tau)$ by continuously modifying τ. Only the projection of the tangent vector to τ is used for the prediction step.

The algorithm can indeed be quite slow, requiring many iterations. However, since $\mathbf{H} = 0$ is solved with an iterative gradient-type algorithm, which enjoys good contracting properties, the correction steps are rather fast.

III.D. Algorithm

We describe the basic steps of the algorithm.

Step 0. Initialization
Using a nominal control $\mathbf{u}(t)$, compute the state trajectory $\mathbf{x}(t)$. Put $\tau = 0$.

Step 1. Computation of the sensitivity matrices
Backward Integration of Eqs. (23, 24, 25).

Step 2. Computation of the updates
With sensitivity matrices $\mathbf{R}, \mathbf{K}, \mathbf{Q}$, compute the control update $\boldsymbol{\delta}\mathbf{u}$ using Eqs. (19, 20, 21). From the sensitivity matrix $\mathbf{Q}$, compute the update on the Lagrange multipliers for the constraints.

Step 3. Evaluate the values of the new constraints.
Integrate the dynamical equations with control $\mathbf{u} + \boldsymbol{\delta}\mathbf{u}$ and for the current value of τ. Then, evaluate the constraints $\boldsymbol{\psi}(\mathbf{x}; t_f)$.

Step 4. Computation of the tangent vector $T(\mathbf{H}(\mathbf{z},\tau))$.
If $\|\boldsymbol{\psi}\| \leq \epsilon$, compute the tangent vector $T(\mathbf{H}(\mathbf{z},\tau))$ as follows:

- Compute the sensitivity of the constraints $\boldsymbol{\psi}$ with respect to the initial costate vector $\boldsymbol{\lambda}_0$ as with Eq. (31).
- Using a SVD, compute the tangent vector in the s direction to the curve of $\mathbf{H}$.
- For a fixed step Δs, compute the change in τ for the next iteration using Eq. (36).

Otherwise, go to step 2.

Step 5. Test of termination
If $\tau = 1$, stop. Otherwise go to step 1.

Steps 2 and 3 provide a correction of the control in a direction to the curve. If the change in τ is small enough from one iteration to the next, steps 2 and 3 perform an exact correction back onto the curve of $\mathbf{H}$. Step 4 must be seen as an estimation step where an new initial guess control is set up for the next iteration.

IV. Applications

IV.A. Orbit Transfer Problem

To illustrate the homotopy method in the gradient based algorithm, a simple academic orbit transfer problem is considered. The dynamics are given by

$$\dot{r} = v_r \tag{38}$$

$$\dot{v_r} = \frac{v_\theta^2}{r} - \frac{\mu_{sun}}{r^2} + A\sin(u) \tag{39}$$

$$\dot{v_\theta} = -\frac{v_r v_\theta}{r} + A\cos(u) \tag{40}$$

where r is the position radius, v_r and v_θ are respectively the radial and ortho-radial velocities. A is the thrust amplitude that evolves with the spacecraft mass, so time t assuming a full thrust at all time (the mass equation can thus be removed from the dynamical system). u is the steering control.

The problem is thus to find the steering control u that drives the spacecraft to satisfy the following terminal constraints at terminal time t_f,

$$\boldsymbol{\psi}(\mathbf{x}; t_f) = \begin{bmatrix} v_r(t_f) \\ v_\theta(t_f) - \frac{\mu_{sun}}{\sqrt{r}} \\ \lambda_r(t_f) - \frac{\lambda_{v_\tau}(t_f)}{2r\sqrt{r}} + 1 \end{bmatrix} \tag{41}$$

writing the costate vector $\boldsymbol{\lambda} = [\lambda_r, \lambda_{v_\tau}, \lambda_{v_r}]^T$. With the objective of maximizing the final radius $r(t_f)$, these constraints place the spacecraft on the highest circular orbit for the given time of flight. The initial conditions are a position on a circular orbit.

The homotopy parameter is the gravitational parameter $\tau \equiv \mu_{sun}$ in normalized units ($\mu_{sun} = 1$). Note then, that as the gravitational parameter changes, both the initial conditions and the terminal constraints change with respect to τ. The homotopy map used is

$$\mathbf{H}(\mathbf{x}, \tau) = \boldsymbol{\psi}(\mathbf{x}) - (1 - \tau)\mathbf{e} \tag{42}$$

$$\mathbf{e} = \boldsymbol{\psi}(\mathbf{x}^0) \tag{43}$$

This homotopy map allows to set any initial guess as initial solution of the homotopy problem. The initial guess control, for $\tau = 0$, is piecewise linear with a discontinuity near half time.

Figure 2 shows the plots of the different solutions when continuing τ. The case for $\tau = 0$ is very significant because it does not include any central force field. One should wonder though if the dynamical model is appropriate and a Cartesian model may be better. The final solution, for $\tau = 1$ is a feasible solution of the problem in the sense that is satisfies the constraints. However, as the optimality conditions have been neglected during the path following, this solution may not be optimal and further refinement may be necessary. Although, this solution is a very good initial guess to the optimal control problem.

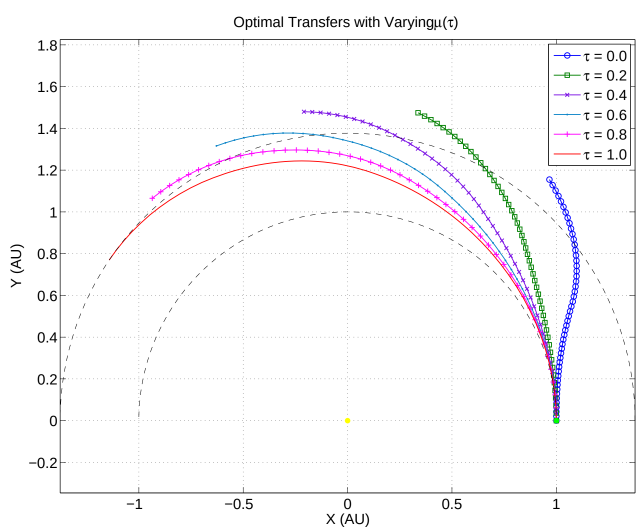

Figure 2. Iterative solutions of the gradient homotopy method for the simple orbital transfer example

Figure 3 pictures the iterations of the different solutions when continuing τ. It can be notice that as μ becomes larger, the continuation step increase and the resolution becomes easier. Still, the number of iterations take to solve the problem with homotopy is larger than for the case without homotopy, but in the former case the user does not have to provide a good initial guess, which is of significant importance for more difficult dynamical problems.

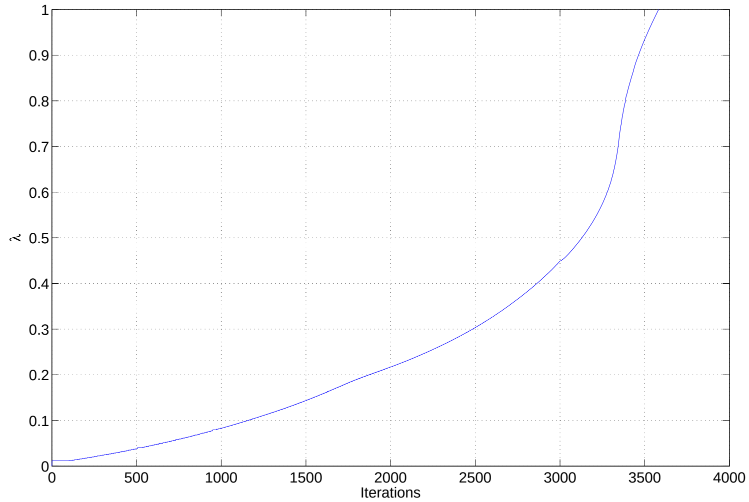

Figure 3. Iterations of the gradient homotopy method for the simple orbital transfer example

IV.B. Restricted Three Body Problem, Transfer to a DRO

The low-thrust circular restricted three-body problem (CRTBP) is studied. The spacecraft departs from an initial Moon orbit and the target is a distant retrograde orbit (DRO). DROs are specific periodic solutions of the CRTBP[25], and they are known for their long-term stability and the low fuel requirement for transferring to them. Their existence is only due to the third body, and they have been widely studied in the scope of planar CRTBPs[26, 25, 27]. In particular, Hénon points out that DRO can exist at very large distance from the second (smaller) body as long as the Hill's problem appropriately approximates the physical problem. They are well suited for the definition of science orbits in very chaotic environments. For instance, few studies have been done for Jupiter mission around Europa[28, 29].

Let's consider the Earth-Moon system, with gravity ratio $\mu = 0.01215$. A circular orbit is assumed and the Earth-Moon distance is approximately to $384400km$. Note in this configuration that $x_{earth} = [\mu, 0, 0]$ and $x_{moon} = [\mu - 1, 0, 0]$. The homotopy problem considers a variation of the thrust amplitude. The rationale is that with high thrust the dynamical problem becomes more controllable, the control structure is simpler and thus convergence is easier. The dynamics are described by

$$\ddot{x} - 2\dot{y} = T_x + \frac{\partial \Omega}{\partial x} \tag{44}$$

$$\ddot{y} + 2\dot{x} = T_y + \frac{\partial \Omega}{\partial y} \tag{45}$$

$$\ddot{z} = T_z + \frac{\partial \Omega}{\partial z} \tag{46}$$

with the auxiliary function

$$\Omega = \frac{x^2 + y^2}{2} + \frac{1 - \mu}{r_{earth}} + \frac{\mu}{r_{moon}} \tag{47}$$

and where $[T_x, T_y, T_z]$ defines the thrust vector. The thrust vector is bounded,

$$|T| \leq T_{max} \tag{48}$$

The following homotopy on the thrust maximum amplitude is considered

$$T_{max} = \tau T^f_{max} + (1 - \tau) T^0_{max} \tag{49}$$

The solution for $\tau = 1$ is the solution of the original problem ($T^0_{max} = 10 T^f_{max}$). The terminal constraints are given by a initial point of the target DRO. Thus,

$$\psi(\mathbf{x}; t_f) = \begin{bmatrix} x(t_f) - x^0_{DRO} \\ y(t_f) \\ z(t_f) \\ \dot{x}(t_f) \\ \dot{y}(t_f) - \dot{y}^0_{DRO} \\ \dot{z}(t_f) \end{bmatrix} \tag{50}$$

Eventually, the time of flight is fixed and the minimum mass solution is sought.

The spacecraft engine has a specific impulse of 2500 s and a nominal thrust amplitude of $T^f_{max} = 0.3$ N. With the homotopy continuation, the thrust amplitude will vary up to double this nominal value. The spacecraft initial mass is 800 kg. The transfer time from a 20000 km circular orbit around the Moon to the 100000 km DRO around the Moon is fixed to be 30 $days$. The entry points of DRO (e.g. vertical velocity) for different distances are summarized up in table 1. The DRO considered lies in the XY-plane of the CRTBP.

Table 1. DRO initial point for different value of μ

μ	x^0_{DRO}	$\dot{y}^0_{DRO}$
0.01215	10000 km	−723.63 m/s
0.01215	50000 km	−483.69 m/s
0.01215	100000 km	−651.57 m/s

Continuation methods have been used for the CRTBP problem for finding the DRO itself, and examples can be found in Ref. [30]. Although, for finding the optimal solution, the usual approaches consider the continuation from the impulsive solution to the low-thrust solution. Other continuation schemes could have been tried, for instance it would be interesting to vary the gravity ratio μ thus transitioning between a two-body problem solution to a three body problem solution. This will be the subject of future work.

Figure 4 depicts the initial and final optimal trajectory, for $\tau = 1$, bearing in mind that the initial solution does not need to be optimal, only feasibly. It was indeed difficult to get a saturating control for the initial guess with high-thrust.

The final mass is 776kg. On Fig. 4 what should be observed is that both the initial and final trajectory are in the plane. It also appears that the main features of the solution remain during the continuation. Indeed, this is quite fortunate as a lower thrust would have make this solution trajectory infeasible.

V. Conclusions

A second order gradient method has been extended with a homotopy technique to solve space trajectory optimal control problem. The homotopy technique reduces the need of good initial guess for the optimization procedure. This approach also shows that it is easier to find solutions that better exploit the complex dynamics of the problem. The overall algorithm proves to be robust but of slow convergence. The method is demonstrated on examples with varying gravity physical parameters.

An attempt is also made for the automatic finding of swing-bys. The approach for swing-by design remains to be made consistent and tested on a wider range of mission profiles. The lack of theory on local optimality of this difficult optimal control problem prevents any direct conclusions.

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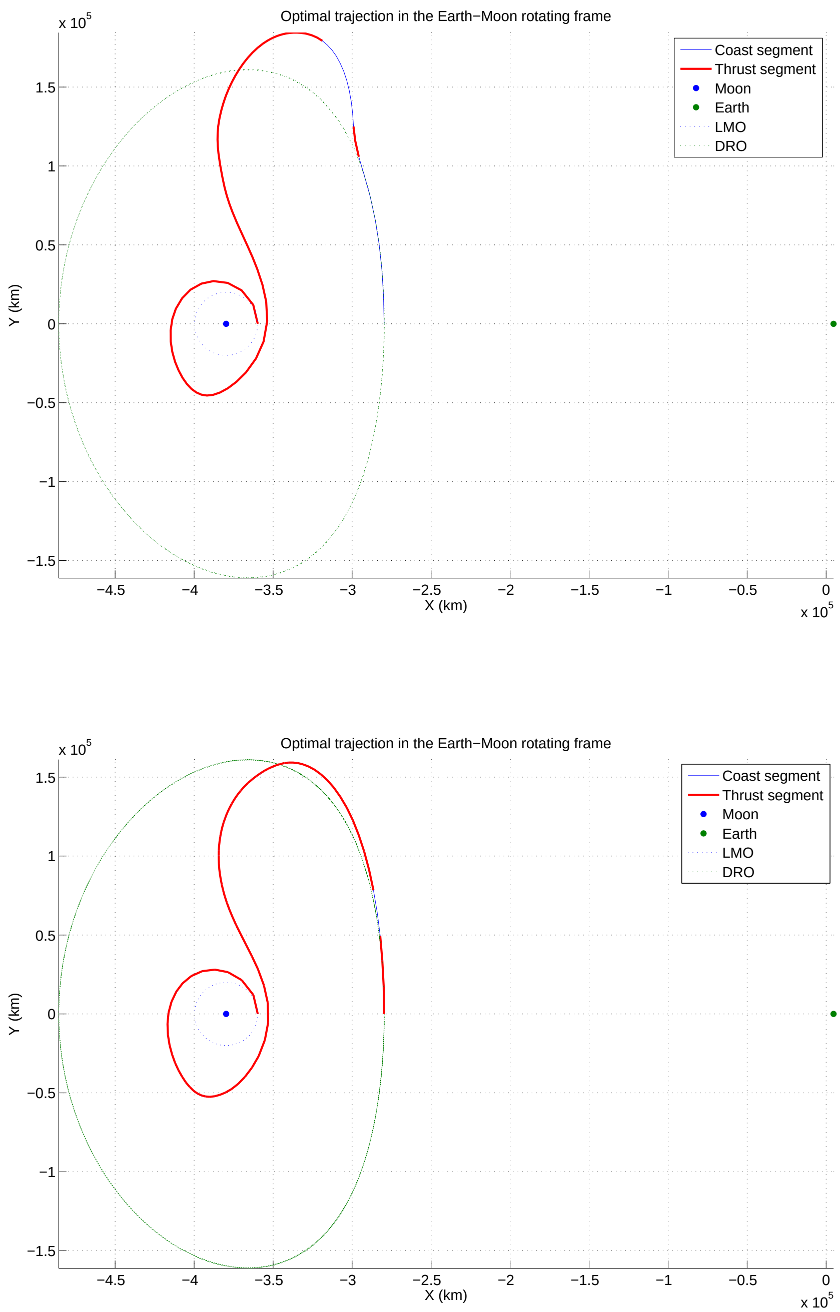

Figure 4. Optimal solution for the transfer to the DRO, with high thrust (top) and final low thrust (bottom).

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