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Matter couplings in Hořava–Lifshitz theories and their cosmological applications

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Abstract
In this paper, the issue of how to introduce matter in Hořava–Lifshitz theories of gravity is addressed. This is a key point in order to complete the proper definition of these theories and, more importantly, to study their possible phenomenological implications. As is well known, in Hořava–Lifshitz gravity, the breakdown of Lorentz invariance invalidates the usual notion of minimally coupled matter. Two different approaches to bypass this problem are described here. One is based on a Kaluza–Klein reinterpretation of the 3+1 decomposition of the gravity degrees of freedom, which naturally leads to a definition of a U(1) gauge symmetry and, hence, to a new type of minimal coupling. The other approach relies on a midi-superspace formalism and the subsequent parametrization of the matter stress–energy tensor in terms of deep infrared variables. Using the last option, the phase space of Hořava–Lifshitz cosmology in the presence of general matter couplings is studied. It is found, in particular, that the equation of state of the effective matter may be very different from the actual matter one, owing to the nonlinear interactions which exist between matter and gravity.

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1. Introduction

Recently, Hořava made a proposal for an ultraviolet completion of general relativity (GR), normally referred to as Hořava–Lifshitz (HL) gravity [1], due to Hořava's initial inspiration on the Lifshitz theory in solid-state physics. The salient characteristic of the HL proposal is that it seems to be renormalizable, at least at the level of power counting. This ultraviolet behavior is obtained by introducing irrelevant operators that explicitly break Lorentz invariance but ameliorate the ultraviolet divergences. On the other hand, Lorentz invariance is expected to be recovered at low energies, as an accidental symmetry of the theory (see [2] for experimental constraints on gravity theories breaking Lorentz invariance and [3] for one possible mechanism, based on supersymmetry, that may lead to the implementation of these constraints).

The original HL proposal has evolved in many aspects and we count nowadays numerous sophisticated versions. In these, new terms have been added to the original Lagrangian, with the idea to generalize the proposal to make it more viable from the phenomenological perspective (see [4]) and to cure the so-called strongly coupled problem ([5, 6] and references therein), via the introduction of new terms [7]. Also, attempts to further generalize this theory at the action level have been undertaken in [8]. Although, as we write this paper, the consistency of the theory and its phenomenological implications still remain uncertain, it seems clear that the above extensions deserve a careful analysis.

An important feature of the original HL theory and its modifications is the breaking of diffeomorphism invariance (Diffi) due to the introduction of precisely those irrelevant operators that cure the UV regime. The lower number of symmetries in the theory, as compared to GR, produces the collateral effect that one looses the notion of 'minimal coupling' between matter and gravity. Another equivalent point of view (at least at the classical level) comes from the covariant formulation of [6, 9] where a Stuckelberg extra scalar degree of freedom over the metric field has to be introduced that may couple to matter in many different ways. In any case, for both formulations (the covariant and the non-covariant one), we have in principle no arguments to choose a particular type of coupling from amongst the most general family of couplings between the gravity and matter sectors. There is very little work in this regard in the literature. For example, in [10] minimal coupling is just assumed, to make contact with GR easier. Other options where some particular couplings have been considered can be found, for example, in [11].

In this paper, two different viable ways to approach this important problem are considered. One of them is a general framework that will teach us how to incorporate, in an educated manner, our ignorance on couplings between matter and gravity. In fact, this method can be used mostly in cosmology, but also in black-hole physics and other general situations in which certain amount of spacetime symmetries are assumed. Here, we will import points of view and basic methodology from midi-super-space approaches and the 3+1 decomposition. The main idea is to parameterize the total four-dimensional energy tensor $\mathcal{T}_{\mu\nu}$ in terms of the deep IR energy variables, thus obtaining a formal expansion where the IR limit corresponds to the usual GR stress–energy tensor and higher order terms correspond to the particular modifications introduced by the HL theory. Since these IR variables represent well-known matter that we see in our laboratories (like for example density and pressure (ρ, p)), they should satisfy the usual equation of state and conservation laws, since the theory is assumed to recover Diffi at low energies. Our other approach is based on a reinterpretation of the 3+1 decomposition as a form of Kaluza–Klein-dimensional reduction, where we still have an untouched three-dimensional Diffi. Then, use is made of the fact that electro-magnetic duality in three dimensions relates one-forms to two-forms, such that we can translate the couplings of matter with the shift N into a $U(1)$ gauge field coupling to charged matter. At this point, we recover a sequence of minimal coupling based on the gravitational $U(1)$ gauge theory. Obviously, this $U(1)$-symmetry is only relevant for the matter sector that couples to N and represents, therefore, only a partial solution to the general problem.

After defining the above frameworks, we proceed to apply these ideas to cosmological scenarios. Much research on this particular aspect of HL gravity has been done in the last two years [12]. Here, in order to address the rather involved issue of studying the cosmological phase space of non-minimally coupled HL gravity, we will borrow, as we already did in a previous work [13], specific techniques from the field of dynamical systems that are frequently used on more canonical studies applied to diverse types of cosmologies (see [14] for a thoughtful introduction to the technique). Another example of this kind of analysis of HL gravity is given in [15]. Later in this paper, we specifically study the *cosmological phase space* of the HL model with our new matter couplings. Our current investigation is focused on the introduction of non-standard couplings between matter and gravity and should be understood as a genuine extension of our previous work on this subject [13] to the cases of real phenomenological interest.

The paper is organized as follows. In section 2, we present a short overview of the relevant modification of the HL theory we are considering, namely the inclusion of a minimal potential defined in [4], and the so-called 'healthy extension' of [7], which presents a whole family of new terms related to the lapse function N. In section 3, we present our framework to study matter couplings, while in section 4, we describe an application of the above frameworks to cosmological scenarios with our generalized matter couplings. We are then able to characterize the classical phase space, discussing its structure in depth, in particular all the fixed points and their nature as repellers or attractors in the theory with matter couplings, in each of the corresponding cases. In section 5, we summarize the results obtained, giving some perspectives for further work.

2. HL without matter

In HL gravity, the gravitational dynamical variables are defined to be the lapse N, the shift N_i and the space metric g_{ij}, Latin indices running from 1 to 3. The spacetime metric is defined using the Arnowitt–Deser–Misner (ADM) slicing of spacetime, as

$$\mathrm{d}s^2 = h_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -N^2\,\mathrm{d}t^2 + g_{ij}(\mathrm{d}x^i + N^i)(\mathrm{d}x^j + N^j)\ , \tag{1}$$

where $N^i = g^{ij}N_j$, as usual. The action S is written in terms of geometric objects, covariant under 3d-diffeomorphisms, characteristics of the ADM construction, like the 3d-covariant derivative ∇_i, the spatial curvature tensor R_{ijkl} and the extrinsic curvature K_{ij}. They are defined as follows:

$$R^i{}_{jkl} = W^i{}_{jl,k} - W^i{}_{jk,l} + W^m{}_{jl}W^i{}_{km} - W^n{}_{jk}W^i{}_{lm}\ , \tag{2}$$

where $W^i{}_{jl}$ are the Christoffel symbols (symmetric in the lower indices), given by

$$W^i_{jl} = \tfrac{1}{2}g^{im}(g_{jm,l} + g_{ml,j} - g_{jl,m}). \tag{3}$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices of the curvature tensor,

$$R_{ij} = g^{kl}R_{ikjl} \quad \text{and} \quad R = g^{ij}R_{ij}\ , \tag{4}$$

while the extrinsic curvature is defined as

$$K_{ij} = \frac{1}{2N}(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i), \tag{5}$$

where the dot stands for time derivative.

In terms of the above tensor fields, the HL action can be written as

$$S = \int \mathrm{d}t\,\mathrm{d}x^3 N\sqrt{g}(\mathcal{L}_{\text{kinetic}} - \mathcal{L}_{\text{potential}} + \mathcal{L}_{\text{matter}}), \tag{6}$$

the kinetic term being universally given by

$$\mathcal{L}_{\rm kinetic} = \alpha(K_{ij}K^{ij} - \lambda K^2)\,, \tag{7}$$

and with α and λ playing the role of coupling constants. Originally, the potential term was a generic function of R_{ijkl} and ∇_i but in [7], it was realized that this generic function should also depend on $a_i = \nabla_i \ln(N)$.

The action generically breaks covariance down to the subgroup of three-dimensional diffeomorphisms and time reparametrization, i.e. $x \to \tilde{x}(t,x)$ and $t \to \tilde{t}(t)$. Assigning dimension -1 to space and dimension -3 to time, it can be seen that it is enough to restrict the potential to be made out of operators up to dimension 6, in order to obtain a power-counting renormalizable theory.

Here, we will work with a potential which corresponds to the more general choice available, composed by the potential defined in [4] (the SVW case), which depends on R_{ijkl} and ∇_i,

$$\begin{aligned}\mathcal{L}_{\rm potential-SVW} = {} & \beta_8\nabla_i R_{jk}\nabla^i R^{jk} + \beta_7 R\nabla^2 R + \beta_6 R^i_j R^j_k R^k_i + \beta_5 R(R_{jk}R^{jk}) \\ & + \beta_4 R^3 + \beta_3 R_{jk}R^{jk} + \beta_2 R^2 + \beta_1 R + \beta_0,\end{aligned} \tag{8}$$

and with the addition of all the general terms as suggested in [7]. We collect all these terms in an implicit form,

$$\mathcal{L}_{{\rm potential}-a_i} = \sum \gamma_n O^n(a_i, \nabla_j, R_{ijkl}), \tag{9}$$

where O^n are general operators of maximum dimension 6 and γ_n are the corresponding coupling constants.

In our calculation, we have worked out all the independent terms of these operators and have chosen to display only, as the representative operator for each class, the ones with less derivatives acting on a single a_i[4]. We therefore obtain
at order 2,

$$\gamma_0 R + \gamma_1 a^2; \tag{10}$$

at order 4,

$$\gamma_3 a^4 + \gamma_4 a^2\nabla_i a^i + \gamma_5\nabla_i a^i\nabla_j a^j + \gamma_6 a^2 R + \gamma_7 a^i a^j R_{ij} + \gamma_8\nabla^2 R + \gamma_9 R_{ij}R^{ij} + \gamma_{10}R^2; \tag{11}$$

and at order 6,

$$\begin{aligned}&\gamma_{11}a^6 + \gamma_{12}a^4 R + \gamma_{13}a^2 a_i a_j R^{ij} + \gamma_{14}a^2 R^2 + \gamma_{15}a_i a_j R^{ij} R + \gamma_{16}a_i a_k R^{ij} R^k_j + \gamma_{17}R^3 \\ &\quad + \gamma_{18}R_{ki}R^{ij}R^k_j + \gamma_{19}R R^{ij}R_{ij} + \gamma_{20}a^2 R^{ij}R_{ij} + \gamma_{21}a^4\nabla_i a^i + \gamma_{22}a^2\nabla_i a^i\nabla_j a^j \\ &\quad + \gamma_{23}\nabla_i a^i\nabla_j a^j_k a^k + \gamma_{24}\nabla_i a^j\nabla_j a^i\nabla_k a^k + \gamma_{25}\nabla_i a^j\nabla_k a^i\nabla^k a_j + \gamma_{26}a^2 a^k\nabla_k R \\ &\quad + \gamma_{27}a^i a^k\nabla_k\nabla_i R + \gamma_{28}a^2\nabla^2 R + \gamma_{29}\nabla^2\nabla^2 R + \gamma_{30}a^i a^j R_{ij}\nabla_k a^k + \gamma_{31}a_i\nabla R^i\nabla_k a^k \\ &\quad + \gamma_{32}a^j R_{ik}\nabla^i R^k_j + \gamma_{33}\nabla^j R_{ik}\nabla^i R^k_j + \gamma_{34}R\nabla^2 R + \gamma_{35}\nabla^i R\nabla_i R + \gamma_{36}R^{ij}\nabla_i\nabla_j R.\end{aligned} \tag{12}$$

The above list includes previous terms of the SW-potential and a number of new terms due to the appearance of the new field a. Note that we have defined the potentials such that they have to be multiplied by $N\sqrt{g}$, which explicitly contains N. Therefore, even though some of the new terms in the potential do not exhibit an explicit coupling with a_i, they are *not* equivalent to any term of the SVW-potential.

[4] This is an arbitrary basis that, nevertheless, fixes our conventions. See also [17] for another expansion in a different basis.

At this point, other phenomenological and theoretical considerations may help us constrain the range of values the different couplings should take. For example, in [16], it was found that ghost instabilities are present if $\lambda \in (1/3, 1)$, that the cosmological constant is negative for the detailed balance potential, $\alpha > 0$, and so on. Here, we will constrain as little as possible the different ranges of values on each coupling constant to see how much information comes out of the dynamical system approach itself. Then, we will add this information to the constraints arising from other considerations, to finally obtain the most promising form of the potential. In particular, we will take λ different from 1/3 (corresponding to the scale-invariant case) as the only limitation on its range.

3. Mater couplings

Once we have explicitly defined the extension of the HL theory we will be working with—at least what concerns its gravity sector—how matter is to be coupled to gravity. As we mentioned in the introduction, due to the reduction of the symmetries present in the theory, we have no longer a valid argument to define a minimal coupling. In fact, this is more dangerous than what it may naively seem, since, for example, different particles will have in general different dispersion relations, depending on their couplings with the gravity sector; also, since we have more geometric invariants, there are many more ways to construct couplings to matter. We postpone this line of thought to future work, to focus here just on the simplest problem of parameterizing the possible form of all these different couplings in terms of physical quantities.

3.1. Midi-superspace approach

The first approach is based on two main assumptions. First, in the deep IR regime, one should recover diffeomorphism invariance and hence, the corresponding IR stress–energy tensor $T_{\mu\nu}$ should be divergence free. Using the above, we can formally expand the full stress–energy tensor, $\mathcal{T}_{\mu\nu}$, in terms of physical observables, defined in the deep IR regime alone. Second, we assume that we have at disposal a considerable number of symmetries, like in cosmological models or black hole physics, which allow us to write $\mathcal{T}_{\mu\nu}$ in terms of just a few variables, as the energy density ρ, the pressure p, the fluid velocity v, etc. The above assumptions imply that we can write

$$\mathcal{T}_{\mu\nu} = T_{\mu\nu}(\rho, p, \ldots) + \Delta T_{\mu\nu}(\rho, p, \ldots), \tag{13}$$

where $\Delta T_{\mu\nu}(\rho, p, \ldots)$ is the leftover contribution, made out of irrelevant operators and other such terms which will anyway decouple at low energies. The above expansion can be understood as a sort of derivative expansion in the gravitational coupling with matter fields. From the point of view of the covariant formulation, where a Stuckelberg field ϕ is added on top of the metric g, what we are doing here is to separate the matter Lagrangian $\mathcal{L}_{\text{matter}}(\psi)$, where ψ represents the matter fields, into a part which is minimally coupled $\mathcal{L}_{\min}(g; \psi)$ and the rest of it, $\mathcal{L}_{\text{non-min}}(\phi, g; \psi)$, which generically has couplings to ϕ and g, e.g.,

$$\mathcal{L}_{\text{matter}}(\psi) = \mathcal{L}_{\min}(\psi; g) + \mathcal{L}_{\text{non-min}}(\psi, g; \phi). \tag{14}$$

Using the above ideas, we can describe our HL theory in terms of a Lagrangian formulation where the gravity sector is given in terms of midi-superspace variables, while the matter sector is described in terms of hydrodynamic variables, like (ρ, p). For example, consider the case of the celebrated FRW ansatz. Here, due to the symmetries imposed, $\mathcal{T}_{\mu\nu}$ depends on two variables (ρ, p) only and it can be written so that it is diagonal. Then, our previous considerations translate into the following expansion:

$$\mathcal{T}_{00} = \rho^T = \rho + \delta_0 p + \delta_1 p^2 + \delta_2 R p + \delta_3 R p^2 + \delta_4 R^2 p + \delta_5 p^3 + \delta_6 \rho, \tag{15}$$

$$\frac{g^{ij}\mathcal{T}_{ij}}{3} = p^T = p + \eta_0 p + \eta_1 p^2 + \eta_2 R p + \eta_3 R p^2 + \eta_4 R^2 p + \delta_5 p^3 + \eta_6 \rho, \tag{16}$$

where, for consistency with the gravity sector, we have limited the expansion to operators up to order 6. Note that here (ρ, p) represent standard matter and, therefore, they fulfill the usual linear relations

$$p = w\rho. \tag{17}$$

Moreover, owing to the non-minimal couplings to gravity, the above equation of state gives rise to nonlinear relations for the effective total energy density and pressure (ρ^T, p^T). These equations, together with the equation of state, will define the type of fluid we can consider in our HL cosmology where matter is coupled to gravity in the most generic form.

Coming back to the general case, as in GR, minimization of the action S, upon variation of the metric in the pure gravity sector, defines the two-index tensor $H\!L_{\mu\nu}$,

$$H\!L_{\mu\nu} = \frac{\delta}{\sqrt{-h}\,\delta h^{\mu\nu}}\left(\int \mathrm{d}t\,\mathrm{d}x^3 N\sqrt{g}(\mathcal{L}_{\text{kinetic}} - \mathcal{L}_{\text{potential}})\right). \tag{18}$$

This tensor can also be decomposed into an IR part, corresponding precisely to the Einstein tensor $G_{\mu\nu}$, plus a leftover, characteristic of the HL theory, which we write as

$$H\!L_{\mu\nu} = \frac{1}{2\kappa^2}\left(G_{\mu\nu} + \Delta H_{\mu\nu}\right), \tag{19}$$

where κ^2 is the gravitational coupling constant (in natural units $\kappa^2 = 8\pi G_N$). Therefore, the form of the field equations obtained by minimizing the HL action with respect to the metric is

$$G_{\mu\nu} + \Delta H\!L_{\mu\nu} = \kappa^2\left(T_{\mu\nu} + \Delta T_{\mu\nu}\right). \tag{20}$$

At this point, we still have to add a last constraint, which comes from taking the 4d divergence to the gravitational field equations (20) and using the usual Bianchi identities for $G_{\mu\nu}$ and $T_{\mu\nu}$. This yields namely

$${}_4\nabla^{\mu}\left(\Delta H\!L_{\mu\nu}\right) = {}_4\nabla^{\mu}\left(\Delta T_{\mu\nu}\right), \tag{21}$$

where ${}_4\nabla$ is the 4d covariant derivative. It is important to recall that these equations have to be satisfied only on-shell, since they actually come from the field equations.

Summarizing, after the whole derivation has been carried out, our final set of equations, that define our recipe to deal with general couplings of matter to HL gravity, is given by equations (17), (20) and (21). Observe that the above expressions are written in terms of a set of operators of order less than or equal to 6, and are made out of geometric objects, such as the 3d curvature R, the vector a_i and the 3d covariant derivative ∇, and of IR hydrodynamic variables, as for example (ρ, p).

3.2. U(1) gravitational coupling

As we have argued previously, due to the breakdown of diffeomorphism invariance, there is no longer a clean argument to define what will be a minimal coupling between gravity and the matter sectors. One could still try, of course, to use a principle based merely on simplicity, but even then, it is a fact that such a principle will by no means be universal, not to talk on its grounding from pure physical considerations. Instead of following the above line of thought, we will here argue that the remaining symmetries of the theory are still strong enough in order to help us find a clear guiding principle which, under some general assumptions, will deliver a well-defined definition of what a minimal coupling in HL theories of gravity should be.

To illustrate our idea, let us consider the more relevant part of the potential terms in the gravity sector of the HL Lagrangian,

$$\int \mathrm{d}t\, \mathrm{d}x^3 N\sqrt{g}(\alpha_0 R + \alpha_1 a^i a_i). \tag{22}$$

We can always device a conformal transformation in the 3d metric, of the form $g_{ij} = N^{-2}\tilde{g}_{ij}$, such that, in the new frame, the vector a and the 3d curvature tensor $\tilde{R}$ are canonically normalized with respect to the 3d metric $\tilde{g}_{ij}$, i.e.

$$\int \mathrm{d}t\, \mathrm{d}x^3 \sqrt{\tilde{g}}(\alpha_0 \tilde{R} + \gamma \tilde{g}^{ij} a_i a_j), \tag{23}$$

where $\gamma = (\alpha_1 - 2\alpha_0)$ and where we have discarded pure boundary terms for simplicity. In this new frame, a_i is a 3d one-form, which can be transformed into a two-form F_{jk} via Hodge duality. The resulting two-form is naturally described in terms of a one-form gauge potential b_i, which immediately leads to a gravitational U(1) gauge symmetry:

$$a^i = \frac{1}{\sqrt{2^3}}\varepsilon^{ijk}F_{jk}, \qquad F_{ij} = \partial_i b_j - \partial_j b_i, \tag{24}$$

where ε is the 3d Levi–Civita pseudo-tensor in the new frame. The constant factor in the above definition is chosen to have a canonically normalized kinetic term in the corresponding action,

$$\int \mathrm{d}t\, \mathrm{d}x^3 \sqrt{\tilde{g}}\left(\alpha_0 \tilde{R} + \frac{1}{4g^2}\tilde{g}^{ij}\tilde{g}^{kl}F_{ik}F_{jl}\right), \tag{25}$$

where $g^{-2} = \gamma$.

From this point of view, we have a local $U(1)$ symmetry that may be used to define the meaning of our new 'minimal coupling' with this gravity sector. Note that, in order to respect such symmetry, we can only write gauge invariant operators in our matter Lagrangian and, therefore, it is clear that our minimal coupling should be written in terms of covariant $U(1)$ derivatives, of the form

$$D_i = \nabla_i - i b_i. \tag{26}$$

For example, the Lagrangian for a charged scalar field, ψ, minimally coupled to the gravity sector, will be given by

$$\mathcal{L} = (\sqrt{g})\left[\frac{1}{N^4}\partial_t\psi\partial_t\psi^* - \tilde{g}^{ij}D_i\psi D_j\psi^* - V(\psi^2)\right], \tag{27}$$

where, as a complementary principle, we have used the fact that, at low energy, we should necessarily recover the usual meaning of minimal coupling.

The case of vector fields and their possible couplings to gravity is more subtle. A gauge field A can be coupled to a conserved current J in the minimal form as $J^i A_i$. In 3d, we have a topologically conserved charge, made out of our gravitational gauge field b, namely $J^i = \varepsilon^{ijk}F_{jk}$. This current is conserved by the corresponding 3d Bianchi identities and, therefore, opens the possibility of including such a term in the usual gauge-invariant action, producing thereby a sort of Chern–Simons term coupling the matter gauge field to the gravity gauge field.

The above idea is far from having been investigated in detail and deserves a lot more attention. We believe that it is indeed an interesting starting point to study such a complicated subject as is the coupling of matter to HL gravity. Note that, as an outcome of these ideas, we are left with a sort of mixed argument, based on symmetries and the low-energy emergence of 4d diffeomorphism invariance, which still does not have a unique, well-defined, general meaning as a 'minimal coupling' in HL gravity. Nevertheless, we have clearly achieved some relevant improvement, since at least the gravity sector related to all the operators made out of a_i is under control, due to the gravitational $U(1)$ symmetry. It will be very interesting to study the implications of the above ideas to BH physics, where the form of the metric clearly induces the coupling with the a_i vector. Hopefully these studies will appear soon elsewhere. In what follows, we will directly concentrate on cosmological scenarios where, owing to the form of the typical ansatz, the $U(1)$ gravitational symmetry is trivial and we still do not take real advantage of the above mechanism.

4. Non-minimal cosmology

In this section, we apply the above methods to study the impact of non-minimal couplings[5] to HL gravity in cosmological scenarios. We start, as our basic ansatz, from the usual FRW metric corresponding to homogeneous and isotropic spacetime, i.e.

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{28}$$

where γ_{ij} is a maximally symmetric metric of constant curvature $k = (-1, 0, 1)$, and a is the scale factor such that

$$R_{ij} = \frac{k}{a^2}g_{ij}\,, \qquad K_{ij} = -Hg_{ij}\,, \qquad H = \frac{\dot{a}}{a}\,. \tag{29}$$

With these symmetries, the total stress–energy tensor $\mathcal{T}_{\mu\nu}$ can be written as

$$\mathcal{T} = \rho^T\,\mathrm{d}t\otimes\mathrm{d}t + p^T g_{ij}\,\mathrm{d}x^i\otimes\mathrm{d}x^j, \tag{30}$$

where ρ^T and p^T are the effective total energy density and effective total pressure, which can be expanded as in equations (15) and (16).

The gravitational sector is drastically simplified in the FRW ansatz where only the terms corresponding to the coupling constant $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ and β_6 in equation (8) are not identically zero. In particular, this means that all terms related to the field $\partial\ln(N)$ do not contribute. Following the work of [4], we assume that our coordinates are such that $c = 1$[6]. We have also used time re-parametrization plus three-dimensional diffeomorphism invariance, in order to eliminate redundant degrees of freedom.

In this setting, the non-zero components of the HL tensor are

$$H\!L_{00} = \alpha\left[3\left(1-\frac{3\xi}{2}\right)H^2 + \frac{3k}{a^2} - \Lambda - \frac{\chi_3 k^2}{2a^4} - \frac{\chi_4 k}{2a^6}\right], \tag{31}$$

$$H\!L_{ij} = -\alpha g_{ij}\left\{\left(1-\frac{3\xi}{2}\right)[2\dot{H}+3H^2] + \frac{k}{a^2} - \Lambda + \frac{\chi_3 k^2}{6a^4} + \frac{\chi_4 k}{2a^6}\right\}, \tag{32}$$

where we set

$$16\pi G_N = \frac{1}{\alpha}\,, \qquad \Lambda = \frac{\beta_0\alpha^3}{2}\,, \qquad 1 = -\beta_1\alpha^2\,,$$
$$\chi_3 = 12\alpha\,(3\beta_2+\beta_3)\,\alpha^2\,, \qquad \chi_4 = 24\,(9\beta_4+3\beta_5+\beta_6)\,\alpha^4\,. \tag{33}$$

[5] We use this term, in contraposition to the usual minimal coupling in GR, to denote a general coupling of matter to HL gravity.

[6] Choosing $c = 1$ is justified to simplify the notation for the gravitational part of the action. Nevertheless, this velocity does not correspond, in general, to the velocity of propagation of matter radiation deep in the UV (see, for example, [18]).

Therefore, we have[7]

$$\Delta \mathcal{HL}_{00} = -\left(\frac{9}{2}H^2 + \frac{\chi_3 k^2}{2a^4} + \frac{\chi_4 k}{2a^6}\right), \tag{34}$$

$$\Delta \mathcal{HL}_{ij} = g_{ij}\left[\frac{3}{2}\xi(2\dot{H} + 3H^2) - \frac{\chi_3 k^2}{6a^4} - \frac{\chi_4 k}{2a^6}\right], \tag{35}$$

while the constraint equation (21) in this framework can be written as

$$(\Delta \dot{\mathcal{HL}}_{00}) + 3H\left(\Delta \mathcal{HL}_{00} + \frac{1}{3}\Delta \mathcal{HL}^i{}_i\right) = (\Delta \dot{\mathcal{T}}_{00}) + 3H\left(\Delta \mathcal{T}_{00} + \frac{1}{3}\Delta \mathcal{T}^i_i\right). \tag{36}$$

It is possible to understand this equation as a measure of up to which level the failure of diffeomorphism invariance in the gravity sector is transmitted to the matter sector. Nevertheless, for the FRW ansatz, it is not difficult to verify that ${}_4\nabla^{\mu}(\Delta \mathcal{HL}_{\mu\nu}) = 0$, due to the homogeneity of the ansatz and, therefore, we are left with the following pure constraint on the matter sector:

$$(\Delta \dot{\mathcal{T}}_{00}) + 3H\left(\Delta \mathcal{T}_{00} + \tfrac{1}{3}\Delta \mathcal{T}^i{}_i\right) = 0. \tag{37}$$

This non-trivial relation for the matter sector ultimately reduces the number of independent couplings that characterize expansion (15), (16). In fact, one finds the following relation between these couplings:

$$\begin{aligned} &\delta_6 + \delta_0 w = \eta_0 + \eta_6 w, && \delta_1(1+2w) = \eta_1, \\ &\delta_5(2+3w) = \eta_5, && \delta_2(w+2/3) = \eta_2, \\ &\delta_3(5/3+2w) = \eta_3, && \delta_4(w+4/3) = \eta_4, \end{aligned} \tag{38}$$

for general values of w.

The field equations for the gravity sector $\mathcal{HL}_{\mu\nu} = \frac{1}{2}\mathcal{T}_{\mu\nu}$ result into the following expression:

$$\left(1 - \frac{3\xi}{2}\right)H^2 - \frac{\chi_2 k}{6a^2} - \frac{\chi_1}{6} - \frac{\chi_3 k^2}{6a^4} - \frac{\chi_4 k}{6a^6} - \kappa^2\rho^T = 0, \tag{39}$$

which, together with equations (15)–(17), and (38) define our cosmological system. The presence of non-minimal couplings of the matter terms conveys the idea that, because of the Lorentz violation, what drives here the cosmological expansion is no more ρ, but actually ρ^T. As a consequence, depending on the values of the parameters δ, a certain type of matter can rather behave effectively as a fluid with different thermodynamical properties. The natural question being then, whether such behavior can possibly help explaining some important features of the observed Universe. In order to answer it, let us first investigate equation (39) for the two basic classical matter types: dust and radiation.

4.1. The dust case ($w = 0$)

In the case of dust, the last equation given above reduces to

$$\left(1 - \frac{3\xi}{2}\right)H^2 - \frac{\chi_2 k}{6a^2} - \frac{\chi_1}{6} - \frac{\chi_3 k^2}{6a^4} - \frac{\chi_4 k}{6a^6} - \kappa^2(1+\delta_6)\rho = 0, \tag{40}$$

which is exactly the same system as treated in [13] with the only difference that the coupling constant is now modified by the parameter δ_6. That is to say, we now have control on the

[7] One could choose to interpret the last two terms of the equations for $\Delta \mathcal{HL}_{\mu\nu}$, as fluid contributions. Given the properties of standard matter in the model, if m is the exponent of a, the perfect fluid corresponding to these terms will have a barotropic factor $w = -1 - \frac{m}{3}$. Thus, for example, the term a^{-6} corresponds to a stiff ($w = 1$) fluid [13].

way in which matter couples to gravity at the cosmological level. Note however that such a small change can induce a great deal of difference in the behavior of the respective cosmology. For example, in the case $\delta_6 < -1$, ρ^T has a negative coupling constant, which generates an effective dark energy, even if the standard matter has the usual thermodynamical properties. In standard GR, this would lead to an irreparable inconsistency of the theory. However, in HL gravity the presence of the additional terms does *not* exclude such case. Of course, once the coupling constant is set to be negative, it stays so forever, and its effect on the cosmic processes typical of the dust era should be investigated carefully.

4.2. The radiation case ($w = \frac{1}{3}$)

Let us now consider the case of radiation. The cosmological equations read

$$H^2\left(1-\frac{3\xi}{2}\right)-\frac{\delta_5\rho_0^3}{27a^{12}}-\frac{2k\delta_3{\rho_0}^2}{3a^{10}}-\frac{\beta_1}{9a^8}-\frac{\beta_2}{6a^6}-\frac{\beta_3}{6a^4}-\frac{k\chi_2}{6a^2}-\frac{\chi_1}{6}=0, \tag{41}$$

$$\left(\frac{3\xi}{2}-1\right)\dot{H}+\left(\frac{3\xi}{2}-1\right)H^2-\frac{5\delta_5\rho_0^3}{27a^{12}}-\frac{8k\delta_3\rho_0^2}{3a^{10}}-\frac{\beta_1}{3a^8}-\frac{\beta_2}{a^6}-\frac{\beta_3}{6a^4}+\frac{\chi_1}{6}=0, \tag{42}$$

with

$$\beta_1 = \rho_0(12^2k^2\delta_4+\delta_1\rho_0), \tag{43}$$

$$\beta_2 = k^3\chi_4+12k\delta_2\rho_0, \tag{44}$$

$$\beta_3 = k^2\chi_3+2(\delta_0+3\delta_6+3)\rho_0. \tag{45}$$

Differently from the previous situation, here the non-vanishing pressure 'switches on' new terms associated with the additional matter couplings. Looking at the general structure of the equations above, however, it is clear that the effects of these couplings will only be relevant at early times, namely when the scale factor a is particularly small. Thus, we can conclude that the introduction of a full matter coupling will influence the evolution of the early Universe, mainly.

It is also interesting to note that, when radiation dominates, the presence of this coupling induces differences between the HL cosmology and the GR one, even in the case of spatially flat solutions. Such a difference is not present in the dust case.

In order to have a more clear idea of the effects of the matter coupling in equations (41) and (42), we can use the dynamical system approach. Given the high number of degrees of freedom of the system, we will limit ourselves to consider the finite analysis only. Following the method of [13], we define the variables

$$\begin{gathered}
X=\frac{\chi_1}{3(3\xi-2)H^2},\quad Y=\frac{2\delta_5\rho_0^3}{27(3\xi-2)H^2a^{12}},\quad Z=\frac{4k\delta_3\rho_0^2}{3(3\xi-2)H^2a^{10}},\\
R=\frac{\beta_1}{9(3\xi-2)H^2a^8},\quad S=\frac{\beta_2}{3(3\xi-2)H^2a^6},\quad T=\frac{\beta_3}{3(3\xi-2)H^2a^4},\\
K=\frac{k\chi_2}{3(3\xi-2)H^2a^2},
\end{gathered} \tag{46}$$

through which we will characterize the phase space, and a logarithmic time $N = \ln a$ (usually known as e-folding).

Table 1. Coordinates, solutions and stability of all the fixed points of the dynamical system (47). The values of the integration constants are obtained from direct substitution into the cosmological equations (41) and (42). Thus, for example, $\rho = 0$ implies that the integration constant ρ_0 of (50) is found to be zero for (41) and (42) to be satisfied.

Point	Coordinates [X,Y,Z,R,S,T]	Solution	Energy density	k	Stability
$\mathcal{A}$	[0, 0, 0, 0, 0, 0]	$a = a_0(t - t_0)$	$\rho = (t - t_0)^{-4}$	-1	Saddle
$\mathcal{B}$	[−1, 0, 0, 0, 0, 0]	$a = a_0 \exp\left(\sqrt{\frac{\chi 1}{3(2-3\xi)}}\, t\right)$	$\rho = 0$	0	Attractor
$\mathcal{C}$	[0, −1, 0, 0, 0, 0]	$a = a_0(t - t_0)^{1/6}$	$\rho = (t - t_0)^{-2/3}$	0	Repeller
$\mathcal{D}$	[0, 0, −1, 0, 0, 0]	$a = a_0(t - t_0)^{1/5}$	$\rho = (t - t_0)^{-4/5}$	0	Saddle
$\mathcal{E}$	[0, 0, 0, −1, 0, 0]	$a = a_0(t - t_0)^{1/4}$	$\rho = (t - t_0)^{-1}$	0	Saddle
$\mathcal{F}$	[0, 0, 0, 0, −1, 0]	$a = a_0(t - t_0)^{1/3}$	$\rho = (t - t_0)^{-4/3}$	0	Saddle
$\mathcal{G}$	[0, 0, 0, 0, 0, −1]	$a = a_0(t - t_0)^{1/2}$	$\rho = (t - t_0)^{-2}$	0	Saddle

Then, the resulting dynamical system is

$$\begin{aligned} X' &= -2X(3R + 2S + T - X + 5Y + 4Z - 1),\\ Y' &= -2Y(3R + 2S + T - X + 5Y + 4Z + 5),\\ Z' &= -2Z(3R + 2S + T - X + 5Y + 4Z + 4),\\ R' &= -2R(3R + 2S + T - X + 5Y + 4Z + 3), \end{aligned} \tag{47}$$

$$\begin{aligned} S' &= -2S(3R + 2S + T - X + 5Y + 4Z + 2),\\ T' &= -2T(3R + 2S + T - X + 5Y + 4Z + 1),\\ 0 &= 1 + K + R + S + T + X + Y + Z, \end{aligned} \tag{48}$$

where the 'prime' indicates derivative with respect to N. The structure of the system reveals that the phase space is divided into different sectors, delimited by invariant submanifolds, and that, as a consequence, no global attractor can actually exist[8]. Therefore, any orbit that can have physical interest will be realized by only using a restricted set of initial conditions.

The fixed points can be found, as usual, by setting the lhs of the equations equal to zero. Then, the solutions associated with the fixed points can be found using the general expressions

$$\dot{H} = \alpha H^2, \tag{49}$$

$$\begin{aligned} \frac{\dot{\rho}}{\rho} &= 4H = -\frac{3}{\alpha(t - t_0)},\\ \alpha &= -1 + 3R_i + 2S_i + T_i - X_i + 5Y_i + 4Z_i, \end{aligned} \tag{50}$$

where the subscript 'i' represents the value of the corresponding variable at the fixed point. Subsequently, using the Hartman–Großmann theorem, we can investigate their stability. The finite fixed points, their associated solution and their corresponding stability are all summarized in table 1.

As expected, we find fixed points associated with the dominance of the different terms in equations (41) and (42). These points are characterized by the corresponding expansion rates

[8] To be precise, this role could be taken by a point at the origin. However, as we will see, such a point would always be unstable.

of the scale factor and of the dissipation of the energy density. Choosing initial conditions in which all the variables are negative (which correspond to a specific set of constraint for the coupling constants χ_i and δ_i), we will select the sector of the phase space in which the orbits have access to all fixed points. In the same way, cosmic histories that surely avoid any of the states described by one of such points can be selected by modifying the values of the couplings.

Note also that none of the fixed points related to the matter couplings appear to be characterized by an accelerated expansion, regardless of the relative sign of the coefficients of those terms. We will discuss in the conclusions the possible consequences of the presence and nature of these points. Note moreover that, in spite of all these changes, one still has a de Sitter attractor which is generated by the cosmological terms in the gravitational part of the action. Therefore, the introduction of the matter couplings does not, in principle, compromise the formation of a dark energy era, in spite of the nature of the cosmic fluid and of the structure of the non-minimal couplings. This is a quite remarkable result.

Substitution into the cosmological equations reveals that the fixed points $\mathcal{A}$ and $\mathcal{C} - \mathcal{G}$ are not actual solutions of the cosmological equations. However, this is not a serious problem, since all these points are unstable and, therefore, they may only correspond to an approximation of the general solution represented by the full orbits. On the contrary, the point $\mathcal{B}$ corresponds, instead, to an exact solution of the theory.

5. Discussion and conclusions

We have proposed in this paper two different ways to study the coupling of HL theories of gravity with matter. First, we have devised a very natural procedure to study the effect of matter couplings in a HL theory of gravity, based on the imposition of the Bianchi identities on a midi-superspace approach together with the assumption of a well-behaved IR limit. The use of these geometric relations has revealed that, when matter is present, we need to supplement the theory with an additional constraint that ensures the consistency of the gravitational and matter sectors. Second, we have introduced an alternative definition of 'minimal coupling' based on the $U(1)$ gauge symmetry present in a very important sector of the theory. In this way, we are able to couple the matter fields via a gauge-invariant formalism. These couplings are both relevant for BH theories and also for perturbation theory in a cosmological scenario, but irrelevant for the study of cosmological solutions, since in this last case the corresponding gravity sector is identically zero.

Regarding the application of the above ideas to cosmological scenarios, in order to put a first set of constraints on this new version of the theory, we have chosen to analyze its Friedmannian cosmology. The high symmetry of cosmological spacetimes greatly simplifies the equations and has allowed us to introduce, in a simple way, a set of (arbitrary) additional matter couplings. In particular, the constraint simplifies into algebraic relations among the coupling constants of the additional matter terms, which can be used to understand their specific physical role.

A first interesting finding here was that, because of the nature of the constraint, the behavior of the HL Universe is very sensitive to the thermodynamical properties of matter. The cosmological equations are enriched with additional terms and both their structure and their number change with the barotropic factor. This implies that the resulting phenomenology is expected to be quite different from one case to another, when we consider Universes dominated by different types of matter.

In addition, although in principle the new matter couplings influence the entire cosmic history, from the structure of the equations it was plain to conclude that the new terms have more weight at early times. This suggests that the most important deviations from standard GR will be evidenced at an early epoch only. This can have important consequences in what concerns the testability of the theory, because some phases of the early history of the Universe (like the nucleosynthesis epoch or the details of the cosmic microwave background spectrum) are very tightly constrained. However, a detailed testing of the theory in this regime would require, to start, a long and careful study which is out of the purpose of this paper. We leave it for a future work.

In order to examine in more detail the specific phenomenology, we have used dynamical system techniques. In the case of dust, we have found that the resulting model is basically the one already analyzed in [13], with the difference that now the sign and the strength of the matter couplings become free parameters of the theory. However, in the end this fact does not have any influence on the number of the fixed points nor on the solutions associated with them, but only on the set of initial conditions that lead to a certain behavior.

The case of radiation is more complex, because of the additional terms that appear in the cosmological equations. In phase space, such terms are associated with additional fixed points ($\mathcal{A}$, $\mathcal{C} - \mathcal{G}$). If all variables are negative, one can obtain the richest possible behavior for the corresponding cosmology, i.e. orbits that are able to 'touch' the entire set of fixed points (albeit they are all unstable). The presence of such fixed points and, in particular, the existence of the solutions associated with them constitute one of the main results of this paper. In fact these fixed points allow us to draw quantitative conclusions on some of the best studied phenomena of the early Universe, as nucleosynthesis or CMB physics.

It is natural to ask if one can choose the above-mentioned constants in such a way that the specified model develops an early-time accelerated expansion phase which might work as inflation. The answer to that interesting issue is unfortunately negative, at least for the time being. The associated dynamical system gives a clear way to see this. In fact, one can prove that, given a generalized Friedman equation of the form

$$H^2 = \frac{\alpha_1}{a^{m_1}} + \frac{\alpha_2}{a^{m_2}} + \cdots + \frac{\alpha_i}{a^{m_i}}, \tag{51}$$

only terms in which $0 < m_i < 2$ can generate fixed points which may represent accelerated expansion. This is clearly not the case for any of the terms appearing in (40) or in (41). What means that, in the phase space, there is no fixed point which could correspond to an inflationary era. However, the situation is not as bad as it might seem at first sight. It has already been suggested that, within the HL framework, other mechanisms might substitute an inflationary phase [10]. Our results seem to support this last, alternative scenario instead of the classical one. Therefore, in spite of the degrees of freedom added by the additional matter couplings, it is very difficult to reproduce any accelerated expansion and the only way in which one can achieve a dark energy era is to introduce the cosmological constant 'by hand' in the HL potential. This appears as a clear limitation of the theory since it does not seem to provide a natural quantum gravity solution to the observation of dark energy[9].

Anyway, the important aspect to be remarked is that, thanks to a consistent introduction of the matter field, the HL theory in this new form finally becomes directly testable against some of the most accurate data we posses at present, and these data hold the promise to be able to directly verify its validity.

[9] It is worth noting that the results above depend critically on the form of (15) and (16). In our calculation, we have chosen a relatively simple structure for the coupling terms, but, in principle one could consider less obvious combinations. In fact one could even reverse engineer these couplings to try to obtain a more desirable behavior. The resulting terms, however, do not have a simple or elegant structure and, for this reason, they have not been displayed in the text.

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