Lorentz-invariant description of the Feigel Process for the Extraction of Momentum from a vacuum

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Relativistically invariant description of the Feigel process for the extraction of momentum from the vacuum

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Abstract

We analyze the effect predicted by Feigel. The covariant constitutive relation for a moving magnetoelectric medium is derived. The latter is then applied to the analysis of the wave propagation in such a medium. Specifically, we study the reflection and refraction of waves at the boundary. Finally, we construct the energy and momentum of waves in a moving magnetoelectric medium and critically re-evaluate the feasibility of the Feigel effect.

1 Introduction

The structure of classical electrodynamics is well established. In particular, in the generally covariant pre-metric approach to electrodynamics [1, 2, 3, 4, 5], the axioms of electric charge and of magnetic flux conservation manifest themselves in the Maxwell equations for the excitation $H = (\mathcal{D}, \mathcal{H})$ and the field strength $F = (E, B)$, namely $dH = J,\ dF = 0$. These equations should be supplemented by a constitutive law $H = H(F)$. The latter relation contains the crucial information about the underlying physical continuum (i.e.,

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spacetime and/or material medium). Mathematically, this constitutive law arises either from a suitable phenomenological theory of a medium or from the electromagnetic field Lagrangian. It can be a nonlinear or even nonlocal relation between the electromagnetic excitation and the field strength. The constitutive law is called a spacetime relation if it applies to spacetime ("the vacuum") itself.

Among many physical applications of classical electrodynamics, the problem of the interaction of the electromagnetic field with matter occupies a central position. The fundamental question, which arises in this context, is about the definition of the energy and momentum in the possibly moving medium. The discussion of the energy-momentum tensor in macroscopic electrodynamics is quite old. The beginning of this dispute goes back to Minkowski [6], Abraham [7], and Einstein and Laub [8]. Nevertheless, up to now the question was not settled and there is an on-going exchange of conflicting opinions concerning the validity of the Minkowski versus the Abraham energy-momentum tensor. Even experiments were not quite able to make a definite and decisive choice of electromagnetic energy and momentum in material media. A consistent solution of this problem has been recently proposed in [20, 3] in the context of a new axiomatic approach to electrodynamics.

Recently Feigel [9] has studied the dynamics of a dielectric magnetoelectric medium in an external electromagnetic field and predicted that the contributions of the quantum vacuum waves (or "virtual photons") could transfer a nontrivial momentum of matter. In our work, we will reconsider this problem in a covariant framework as developed earlier in [3, 20].

The plan of the study is as follows. In Sec. 2 we give an introduction and a short overview of the Feigel effect. The corresponding theoretical input, which is needed for the discussion and evaluation of this effect, is listed in Sec. 3. Then, in Sec. 4, we construct the relativistic local and linear constitutive relation for a magnetoelectric medium at rest and in motion. Next, in Sec. 5, a general analysis of the wave propagation in magnetoelectric media is given specifying the typical birefringence effects. In Sec. 6, we consider the propagation of vacuum fluctuations in the form of the plane waves *through* a finite magnetoelectric sample and compute the energy and momentum outside the matter. Our conclusions are formulated in Sec. 7.

2 Preliminaries: the Feigel effect

The Feigel effect [9] can be explained in simple terms as follows: Let us consider an isotropic homogeneous medium with the electric and magnetic constants ε, μ. Electromagnetic waves are propagating in such a medium absolutely symmetrically, with the Fresnel equation describing the unique light cone. This is easily derived from the constitutive relations $\mathcal{D} = \varepsilon\varepsilon_0\, E$ and $\mathcal{H} = (\mu\mu_0)^{-1}B$.

However, if a medium is placed in crossed constant external electric and magnetic fields, then it acquires magnetoelectric properties. As a result, we have the *anisotropic* magnetoelectric medium with ε, μ, plus the magnetoelectric (matrix) parameter χ (determined by the external fields) which modifies the constitutive relations to $\mathcal{D} = \varepsilon\varepsilon_0\, E + \chi\, B$ and $\mathcal{H} = (\mu\mu_0)^{-1}B - \chi^{\rm T}E$; here $^{\rm T}$ denotes the transposed matrix.

Accordingly, the wave propagation in such a medium also becomes anisotropic and birefringent, with the wave covectors now belonging to two light cones. Applying this to vacuum waves (or, perhaps, better to say to the "vacuum fluctuations" or "virtual photons") propagating in the magnetoelectric body, Feigel [9] computed the total momentum carried by these waves and concluded that it is non-vanishing. In accordance with this derivation, a body should move with a small but non-negligeable velocity. Earlier the Feigel process was discussed in [10, 11, 12, 13].

The purpose of our study is to re-evaluate the feasibility of the Feigel effect. We will do this on the basis of the covariant premetric formulation of classical electrodynamics [3].

3 Theoretical input

In order to critically evaluate the feasibility of the Feigel process, let us recall what theoretical input is needed for its computation.

- Quantum theory

  Feigel assumed that "virtual photons" or "vacuum waves" of electromagnetic field are moving inside matter. This is an elementary picture which was used previously for the analysis of various classical and quantum effects of the electromagnetic field (such as, for example, the Lamb shift and the Casimir effect). Following Feigel, we will use this assumption in our discussion.

- Constitutive relation for moving media

  The covariant premetric formulation of classical electrodynamics [3] provides a general approach for the derivation of the constitutive relation for an arbitrarily moving material medium. Feigel used the Lorentz (or, more exactly, Galilei) transformation to derive the constitutive law for the moving magnetoelectric matter. However, classical electrodynamics is not, in fact, related to the Lorentz group. Instead, the field equations are *generally covariant*. The technical tool which specifies the motion of matter is the so called foliation structure. By carefully distinguishing the laboratory foliation from the material one and by establishing the link between them, it is possible to derive the correct constitutive relation for any medium moving arbitrarily in spacetime. We will assume that the excitation $(\mathcal{H},\mathcal{D}) = H = H_{ij}\,dx^i dx^j/2$ is a local linear function of field strength $(E,B) = F = F_{ij}\,dx^i dx^j/2$:

  $$H = \kappa(F)\,, \qquad H_{ij} = \frac{1}{2}\,\kappa_{ij}{}^{kl}\,F_{kl}\,. \tag{1}$$

  We will derive the corresponding constitutive tensor κ for moving magnetoelectric matter.

- Relativistic fluid dynamics

  Since the process under consideration predicts a nontrivial dynamics for matter, it is necessary to establish the equations of motion for the magnetoelectric medium. Feigel [9] used a non-relativistic approach. However, the relativistic theory of a moving ideal fluid would be a more robust framework. We develop the consistent variational theory of a relativistic fluid and compare it with the Feigel's derivations.

- Energy-momentum of the electromagnetic field

  Since the Feigel process is about "extracting momentum from vacuum", it is important to know how exactly the electromagnetic field momentum is expressed in terms of the field components and the material parameters. The generally covariant approach provides the momentum as a part of the energy-momentum tensor $T_i{}^j$, with energy density $u = T_0{}^0$, energy flux density $s^a = T_0{}^a$, momentum density $p_a = -T_a{}^0$, and stress $T_a{}^b$.

4 Constitutive relation

Within the axiomatic premetric generally covariant framework [3], the projection technique is used to define the electric and magnetic phenomena in an arbitrarily moving medium. As in [3], we assume that the spacetime is foliated into spatial slices with time σ and transverse vector field n. Then we decompose any form Ψ into a part *longitudinal* with respect to n,

$$^{\perp}\Psi := d\sigma \wedge \Psi_{\perp}\,, \qquad \Psi_{\perp} := n\rfloor\Psi\,, \tag{2}$$

and a part *transversal* with respect to n:

$$\underline{\Psi} := (1\; - \;^{\perp})\Psi = n\rfloor(d\sigma \wedge \Psi)\,, \quad n\rfloor\,\underline{\Psi} \equiv 0\,. \tag{3}$$

Thus, we have a general decomposition $\Psi = \;^{\perp}\Psi + \underline{\Psi} = d\sigma \wedge \Psi_{\perp} + \underline{\Psi}$.

When applying this to the 2-forms H and F, we obtain the magnetic $\mathcal{H}$ and electric $\mathcal{D}$ excitations as longitudinal and transversal parts of H, and, similarly, electric E and magnetic B fields as longitudinal and transversal parts of F, namely

$$H = -\mathcal{H} \wedge d\sigma + \mathcal{D}, \qquad \text{and} \qquad F = E \wedge d\sigma + B. \tag{4}$$

This foliation is called the *laboratory* foliation.

Along with the original κ-tensor (1), it is convenient to introduce an alternative representation of the constitutive tensor:

$$\chi^{ijkl} := \frac{1}{2}\,\epsilon^{ijmn}\,\kappa_{mn}{}^{kl}. \tag{5}$$

Performing a $(1+3)$-decomposition of covariant electrodynamics, as described above, we can write H and F as column 6-vectors with the components built from the magnetic and electric excitation 3-vectors $\mathcal{H}_a, \mathcal{D}^a$, and the electric and magnetic field strengths E_a, B^a, respectively. Then the linear spacetime relation (1) reads:

$$\begin{pmatrix} \mathcal{H}_a \\ \mathcal{D}^a \end{pmatrix} = \begin{pmatrix} \mathcal{C}^b{}_a & \mathcal{B}_{ba} \\ \mathcal{A}^{ba} & \mathcal{D}_b{}^a \end{pmatrix} \begin{pmatrix} -E_b \\ B^b \end{pmatrix}. \tag{6}$$

Here the constitutive tensor is conveniently represented by the 6×6-matrix

$$\kappa_I{}^K = \begin{pmatrix} \mathcal{C}^b{}_a & \mathcal{B}_{ba} \\ \mathcal{A}^{ba} & \mathcal{D}_b{}^a \end{pmatrix}, \qquad \chi^{IK} = \begin{pmatrix} \mathcal{B}_{ab} & \mathcal{D}_a{}^b \\ \mathcal{C}^a{}_b & \mathcal{A}^{ab} \end{pmatrix}. \tag{7}$$

The constitutive 3×3 matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are constructed from the components of the original constitutive tensor as

$$\mathcal{A}^{ba} := \chi^{0a0b}, \qquad \mathcal{B}_{ba} := \frac{1}{4}\,\hat{\epsilon}_{acd}\,\hat{\epsilon}_{bef}\,\chi^{cdef}, \tag{8}$$

$$\mathcal{C}^{a}{}_{b} := \frac{1}{2}\,\hat{\epsilon}_{bcd}\,\chi^{cd0a}, \qquad \mathcal{D}_{a}{}^{b} := \frac{1}{2}\,\hat{\epsilon}_{acd}\,\chi^{0bcd}. \tag{9}$$

If we resolve with respect to χ, we find the inverse formulas

$$\chi^{0a0b} = \mathcal{A}^{ba}, \qquad \chi^{abcd} = \epsilon^{abe}\,\epsilon^{cdf}\,\mathcal{B}_{fe}, \tag{10}$$

$$\chi^{0abc} = \epsilon^{bcd}\,\mathcal{D}_{d}{}^{a}, \qquad \chi^{ab0c} = \epsilon^{abd}\,\mathcal{C}^{c}{}_{d}. \tag{11}$$

In this study, we assume that the skewon and the axion are absent so that the constitutive matrices satisfy $\mathcal{A}^{ab} = \mathcal{A}^{ba}$, $\mathcal{B}_{ab} = \mathcal{B}_{ba}$, and $\mathcal{D}_{b}{}^{a} = \mathcal{C}^{a}{}_{b}$, with $\mathcal{C}^{a}{}_{a} = 0$.

4.1 Magnetoelectric medium at rest

We begin the discussion of magnetoelectric media by recalling that the line element with respect to the laboratory foliation coframe reads

$$ds^2 = N^2\,d\sigma^2 + g_{ab}\,\underline{dx}^a\,\underline{dx}^b = N^2\,d\sigma^2 - {}^{(3)}g_{ab}\,\underline{dx}^a\,\underline{dx}^b. \tag{12}$$

Here $N^2 = \mathbf{g}(n,n)$ is the length of the foliation vector field n, and $\underline{dx}^a = dx^a - n^a\,d\sigma$ is the transversal 3-covector basis, in accordance with the definitions above. The 3-metric ${}^{(3)}g_{ab}$ is the positive definite Riemannian metric on the spatial 3-dimensional slices corresponding to fixed values of the time σ. This metric defines the 3-dimensional Hodge duality operator $\underline{\star}$.

A conventional magnetoelectric medium is characterized by the traceless matrix $\mathcal{C}^{a}{}_{b}$ with 8 independent components, cf. [14]. Its symmetric and antisymmetric pieces $\mathcal{C}^{(ab)}$ and $\mathcal{C}^{[ab]}$ have 5 and 3 independent components, respectively. Since in the setup of the Feigel effect, the magnetoelectric properties are "excited" by means of the external crossed electric and magnetic fields, we expect that $\mathcal{C}^{[ab]} = \epsilon^{abc} m_c$ with the 3-vector m_c proportional to the vector product of external crossed fields, see the experimental results [15]. Hence we introduce the covector (1-form) m which is purely transversal, $n\rfloor m = 0$, i.e., $m = m_a \underline{dx}^a$. This modifies the constitutive relation as follows:

$$\mathcal{D} = \varepsilon\varepsilon_0\varepsilon_g\,{}^\star E + {}^\star B \wedge m, \tag{13}$$

$$\mathcal{H} = \frac{1}{\mu\mu_0\mu_g}\,{}^\star B + {}^\star(E \wedge m). \tag{14}$$

Here $\varepsilon_g = \mu_g = c/N$ are effective electric and magnetic permeabilities of the spacetime. When $m = 0$, we have the isotropic medium with the electric and magnetic constants ε and μ.

It is straightforward to see that the covector m has the dimension of a conductance (inverse resistance), i.e., $[m] = [\lambda_0]$.

4.2 Magnetoelectric medium in motion

The dynamics of a material medium is encoded in the structure of another foliation (τ, u) which is determined by the four-vector field of the velocity u of matter and the proper time coordinate τ. Accordingly, we have to formulate the constitutive law with respect to this, so called material foliation. As a first step, we observe that the relation between the two coframe bases, namely those of the laboratory foliation $(d\sigma, \underline{dx}^a)$ and of the material foliation $(d\tau, \underset{\sim}{dx}^a)$ is as follows

$$\begin{pmatrix} d\sigma \\ \underline{dx}^a \end{pmatrix} = \left(\begin{array}{c|c} \gamma c/N & v_b/(cN) \\ \hline \gamma v^a & \delta^a_b \end{array}\right) \begin{pmatrix} d\tau \\ \underset{\sim}{dx}^b \end{pmatrix}. \tag{15}$$

Here, for the *relative velocity* 3-vector, we introduced the notation

$$v^a := \frac{c}{N}\left(\frac{u^a}{u^{(\sigma)}} - n^a\right), \quad \text{with} \quad \gamma := \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{16}$$

Substituting (15) into (12), we find for the line element in terms of the new variables

$$ds^2 = c^2\,d\tau^2 - \widehat{g}_{ab}\,\underset{\sim}{dx}^a\,\underset{\sim}{dx}^b, \quad \text{where} \quad \widehat{g}_{ab} = {}^{(3)}g_{ab} - \frac{1}{c^2}\,v_a v_b. \tag{17}$$

The metric $\widehat{g}_{ab}$ of the material foliation has the inverse

$$\widehat{g}^{ab} = {}^{(3)}g^{ab} + \frac{\gamma^2}{c^2}\,v^a v^b. \tag{18}$$

For its determinant one finds $(\det \widehat{g}_{ab}) = (\det g_{ab})\,\gamma^{-2}$. We will denote the 3-dimensional Hodge star defined by this metric as $\widehat{{}^*}$.

In order to find the constitutive relation for the for the magnetoelectric moving medium with respect to the laboratory reference frame, we start with the constitutive law with respect to the material foliation

$$\begin{aligned} \mathcal{D}' &= \varepsilon\varepsilon_0\,\widehat{{}^*}E' + \widehat{{}^*}B' \wedge m, && (19)\\ \mathcal{H}' &= \frac{1}{\mu\mu_0}\,\widehat{{}^*}B' + \widehat{{}^*}(E' \wedge m). && (20) \end{aligned}$$

The primes denote the quantities taken with respect to the moving frame. The magnetoelectric covector reads $m = m_a \underset{\sim}{dx}{}^a$.

The constitutive relation (19),(20) can be presented in the equivalent matrix form

$$\begin{pmatrix} \mathcal{H}'_a \\ \mathcal{D}'^a \end{pmatrix} = \lambda \begin{pmatrix} C'^b{}_a & B'_{ab} \\ A'^{ab} & C'^a{}_b \end{pmatrix} \begin{pmatrix} -E'_b \\ B'^b \end{pmatrix} \tag{21}$$

The components of the constitutive matrices read explicitly

$$\begin{aligned} A'^{ab} &= -\frac{n}{c}\frac{\sqrt{g}}{\gamma}\,\widehat{g}^{ab}, \quad B'_{ab} = \frac{c}{n}\frac{\gamma}{\sqrt{g}}\,\widehat{g}_{ab}, && (22)\\ C'^a{}_b &= \widehat{\eta}^{acd}\,\widehat{g}_{bc}\,m_d/\lambda, && (23) \end{aligned}$$

with $\lambda = \sqrt{\frac{\varepsilon\varepsilon_0}{\mu\mu_0}}$, $n := \sqrt{\mu\varepsilon}$, and $\widehat{\eta}^{abc} = \epsilon^{abc}/\sqrt{\det \widehat{g}} = \gamma\epsilon^{abc}/\sqrt{\det g}$.

In order to find the constitutive law in the laboratory frame, we have to perform some very straightforward manipulations in matrix algebra along the lines described in Hehl and Obukhov [3], Sec. D.5.4. Given is the linear transformation of the coframes (15). The corresponding transformation of the 2-form basis (A.1.95) of [3] turns out to be

$$\begin{aligned} P^a{}_b &= \frac{\gamma c}{N}\left(\delta^a_b - \frac{1}{c^2}\,v^a v_b\right), \quad Q_b{}^a = \delta^a_b, \\ Z_{ab} &= -\,\gamma\hat{\epsilon}_{abc}\,v^c, \qquad W^{ab} = \frac{1}{Nc}\,\epsilon^{abc}\,v_c. && (24) \end{aligned}$$

We use these results in (D.5.27)–(D.5.30) of [3]. Then, after a lengthy matrix computation, we obtain from (22)-(23) the constitutive matrices in the laboratory foliation:

$$
\begin{aligned}
A^{ab} &= \frac{1}{1-\frac{v^2}{c^2}}\,\frac{\sqrt{{}^{(3)}g}}{N}\left[{}^{(3)}g^{ab}\left(\frac{v^2}{c^2 n}-n\right)+\frac{1}{c^2}\,v^a v^b\left(n-\frac{1}{n}\right)\right]\\
&\quad+\frac{2}{\sqrt{1-\frac{v^2}{c^2}}}\,\frac{\sqrt{{}^{(3)}g}}{Nc\lambda}\left[{}^{(3)}g^{ab}\,(mv)-v^{(a}m^{b)}\right], \qquad (25)\\
B_{ab} &= \frac{1}{1-\frac{v^2}{c^2}}\,\frac{N}{\sqrt{{}^{(3)}g}}\left[{}^{(3)}g_{ab}\left(\frac{1}{n}-\frac{v^2 n}{c^2}\right)+\frac{1}{c^2}\,v_a v_b\left(n-\frac{1}{n}\right)\right]\\
&\quad+\frac{2}{\sqrt{1-\frac{v^2}{c^2}}}\,\frac{N}{c\lambda\sqrt{{}^{(3)}g}}\left[{}^{(3)}g_{ab}\,(mv)-v_{(a}m_{b)}\right], \qquad (26)\\
C^a{}_b &= \frac{1}{1-\frac{v^2}{c^2}}\left(n-\frac{1}{n}\right){}^{(3)}\eta^{ac}{}_b\,\frac{v_c}{c}\\
&\quad-\frac{1}{\lambda\sqrt{1-\frac{v^2}{c^2}}}\;{}^{(3)}\eta^{ac}{}_b\left[m_c+v_c(mv)/c^2\right]. \qquad (27)
\end{aligned}
$$

Here we denote $(mv) = m_a v^a$ (the raising and lowering of the indices is performed with the help of the spatial metric ${}^{(3)}g_{ab}$).

The resulting constitutive law for the moving magnetoelectric medium reads

$$
\begin{pmatrix} \mathcal{H}_a \\ \mathcal{D}^a \end{pmatrix} = \lambda \begin{pmatrix} C^b{}_a & B_{ab} \\ A^{ab} & C^a{}_b \end{pmatrix} \begin{pmatrix} -E_b \\ B^b \end{pmatrix} \qquad (28)
$$

This constitutive relation is valid on an arbitrary curved spacetime and for an arbitrary motion of the medium. In other words, the constitutive relation (28) is generally covariant (diffeomorphism covariant). Since Feigel [9] studied the proposed effect in the Minkowski spacetime, we eventually will also specify our derivations for this case. Then we have to put $N = c$ and ${}^{(3)}g_{ab} = \delta_{ab} = \mathrm{diag}(1, 1, 1)$. We will assume this from now on.

5 Wave propagation: birefringence in magnetoelectric media

The Fresnel approach (geometric optics) to the wave propagation in media and in spacetime with the general linear constitutive law gives rise to the *extended covariant Fresnel equation* for the wave covector q_i:

$$\mathcal{G}^{ijkl}(\chi)\, q_i q_j q_k q_l = 0\,. \tag{29}$$

Here the fourth order Tamm-Rubilar (TR) tensor density of weight $+1$ is defined by

$$\mathcal{G}^{ijkl}(\chi) := \frac{1}{4!}\,\hat{\epsilon}_{mnpq}\,\hat{\epsilon}_{rstu}\,\chi^{mnr(i}\,\chi^{j|ps|k}\,\chi^{l)qtu}\,. \tag{30}$$

Let us denote the independent components of the TR-tensor (30) as follows:

$$M := \mathcal{G}^{0000} = \det\mathcal{A}\,, \tag{31}$$

$$M^a := 4\,\mathcal{G}^{000a} = -\hat{\epsilon}_{bcd}\left(\mathcal{A}^{ba}\,\mathcal{A}^{ce}\,\mathcal{C}^d{}_e + \mathcal{A}^{ab}\,\mathcal{A}^{ec}\,\mathcal{D}_e{}^d\right)\,, \tag{32}$$

$$\begin{aligned} M^{ab} := 6\,\mathcal{G}^{00ab} &= \frac{1}{2}\,\mathcal{A}^{(ab)}\left[(\mathcal{C}^d{}_d)^2 + (\mathcal{D}_c{}^c)^2 - (\mathcal{C}^c{}_d + \mathcal{D}_d{}^c)(\mathcal{C}^d{}_c + \mathcal{D}_c{}^d)\right] \\ &+ (\mathcal{C}^d{}_c + \mathcal{D}_c{}^d)(\mathcal{A}^{c(a}\mathcal{C}^{b)}{}_d + \mathcal{D}_d{}^{(a}\mathcal{A}^{b)c}) - \mathcal{C}^d{}_d\mathcal{A}^{c(a}\mathcal{C}^{b)}{}_c \\ &- \mathcal{D}_c{}^{(a}\mathcal{A}^{b)c}\mathcal{D}_d{}^d - \mathcal{A}^{dc}\mathcal{C}^{(a}{}_c\mathcal{D}_d{}^{b)} + \left(\mathcal{A}^{(ab)}\mathcal{A}^{dc} - \mathcal{A}^{d(a}\mathcal{A}^{b)c}\right)\mathcal{B}_{dc}, \end{aligned} \tag{33}$$

$$\begin{aligned} M^{abc} := 4\,\mathcal{G}^{0abc} &= \epsilon^{de(c|}\left[\mathcal{B}_{df}(\mathcal{A}^{ab)}\,\mathcal{D}_e{}^f - \mathcal{D}_e{}^a\mathcal{A}^{b)f}\,)\right. \\ &\left.+ \mathcal{B}_{fd}(\mathcal{A}^{ab)}\,\mathcal{C}^f{}_e - \mathcal{A}^{f|a}\mathcal{C}^{b)}{}_e) + \mathcal{C}^a{}_f\,\mathcal{D}_e{}^{b)}\,\mathcal{D}_d{}^f + \mathcal{D}_f{}^a\,\mathcal{C}^{b)}{}_e\,\mathcal{C}^f{}_d\right]\,, \end{aligned} \tag{34}$$

$$M^{abcd} := \mathcal{G}^{abcd} = \epsilon^{ef(c}\epsilon^{|gh|d}\,\mathcal{B}_{hf}\left[\frac{1}{2}\,\mathcal{A}^{ab)}\,\mathcal{B}_{ge} - \mathcal{C}^a{}_e\,\mathcal{D}_g{}^{b)}\right]\,. \tag{35}$$

Then, in $(1+3)$-decomposed form, the extended Fresnel equation (29) reads

$$q_0^4 M + q_0^3 q_a\, M^a + q_0^2 q_a q_b\, M^{ab} + q_0 q_a q_b q_c\, M^{abc} + q_a q_b q_c q_d\, M^{abcd} = 0\,. \tag{36}$$

5.1 Medium at rest

Using the constitutive relation (13), (14) in (31)-(35), we find explicitly:

$$M = -\left(\frac{n}{c}\right)^3, \qquad M^a = 4\left(\frac{n}{c}\right)^2\overline{m}^a, \tag{37}$$

$$M^{ab} = \frac{n}{c}\left[\delta^{ab}(2+\overline{m}^2) - 5\overline{m}^a\overline{m}^b\right], \tag{38}$$

$$M^{abc} = -\,2\delta^{(ab}\overline{m}^{c)}\,(2+\overline{m}^2) + 2\overline{m}^a\overline{m}^b\overline{m}^c, \tag{39}$$

$$M^{abcd} = \frac{c}{n}\left[-\delta^{(ab}\delta^{cd)}(1+\overline{m}^2) + \delta^{(ab}\overline{m}^c\overline{m}^{d)}\right]. \tag{40}$$

Here we introduced the dimensionless magnetoelectric vector $\overline{m}_a := m_a/\lambda$, and we denote $\overline{m}^a = \delta^{ab}\overline{m}_b$ and $\overline{m}^2 = \overline{m}^a\overline{m}_a$.

Substituting this into the Fresnel equation (29), (36), we find the *birefringence* effect: the quartic wave surface is factorized into the product of the two light cones,

$$\mathcal{G}^{ijkl}(\chi)\, q_i q_j q_k q_l = -\, \left(g_1^{ij} q_i q_j\right)\, \left(g_2^{kl} q_k q_l\right) = 0. \tag{41}$$

The two optical metrics depend explicitly on the magnetoelectric properties according to

$$g_1^{ij} = \left(\begin{array}{c|c} \frac{n^2}{c^2} & -\frac{n}{c}\overline{m}^b \\ \hline -\frac{n}{c}\overline{m}^a & -\delta^{ab} \end{array}\right) \tag{42}$$

and

$$g_2^{ij} = \left(\begin{array}{c|c} \frac{n^2}{c^2} & -\frac{n}{c}\overline{m}^b \\ \hline -\frac{n}{c}\overline{m}^a & -\delta^{ab}(1+\overline{m}^2) + \overline{m}^a\overline{m}^b \end{array}\right), \tag{43}$$

respectively. It is interesting that the magnetoelectric vector manifests itself as an effective "rotation" of the spacetime related to the off-diagonal components of the optical metric.

5.2 Medium in motion

In this case, we start from the constitutive relation (19), (20) and (31)-(35) then yields

$$\begin{aligned}
M' &= -\frac{\sqrt{g}}{\gamma}\left(\frac{n}{c}\right)^3, \qquad M'^a = 4\frac{\sqrt{g}}{\gamma}\left(\frac{n}{c}\right)^2 \overline{m}^a, && (44)\\
M'^{ab} &= \frac{\sqrt{g}}{\gamma}\frac{n}{c}\left[\widehat{g}^{ab}(2+\overline{m}^2) - 5\overline{m}^a\overline{m}^b\right], && (45)\\
M'^{abc} &= -2\frac{\sqrt{g}}{\gamma}\widehat{g}^{(ab}\overline{m}^{c)}\left(2+\overline{m}^2\right) + 2\overline{m}^a\overline{m}^b\overline{m}^c, && (46)\\
M'^{abcd} &= \frac{\sqrt{g}}{\gamma}\frac{c}{n}\left[-\widehat{g}^{(ab}\widehat{g}^{cd)}(1+\overline{m}^2) + \widehat{g}^{(ab}\overline{m}^c\overline{m}^{d)}\right]. && (47)
\end{aligned}$$

The primes are denoting the components with respect to the moving material foliation. Moreover, $\overline{m}^a = \widehat{g}^{ab}\overline{m}_b$ and $\overline{m}^2 = \overline{m}^a\overline{m}_a$.

As a result, we find the Fresnel surface, similarly to (41), factorized into the product of the two light cones

$$\mathcal{G}^{ijkl}(\chi)\, q'_i q'_j q'_k q'_l = -\frac{\sqrt{g}}{\gamma}\left(g_1'^{ij} q'_i q'_j\right)\, \left(g_2'^{kl} q'_k q'_l\right) = 0, \tag{48}$$

with the optical metrics given (in the moving reference frame) by

$$g_1'^{ij} = \left(\begin{array}{c|c} \frac{n^2}{c^2} & -\frac{n}{c}\overline{m}^b \\ \hline -\frac{n}{c}\overline{m}^a & -\widehat{g}^{ab} \end{array}\right), \tag{49}$$

and

$$g_2'^{ij} = \left(\begin{array}{c|c} \frac{n^2}{c^2} & -\frac{n}{c}\overline{m}^b \\ \hline -\frac{n}{c}\overline{m}^a & -\widehat{g}^{ab}(1+\overline{m}^2)+\overline{m}^a\overline{m}^b \end{array}\right). \tag{50}$$

Finally, the components of the optical metrics with respect to the laboratory foliation are obtained from

$$g_{1,2}^{ij} = L^i{}_k L^j{}_l \, g_{1,2}'^{kl}, \tag{51}$$

where the matrix L describes the transformation between the coframe bases of the two foliations (15), namely

$$L^i{}_j = \left(\begin{array}{c|c} \gamma c/N & v_b/(cN) \\ \hline \gamma v^a & \delta^a_b \end{array}\right). \tag{52}$$

The same result can be obtained from an alternative (somewhat longer) computation by means of substituting the constitutive matrices (25)-(27) directly into the formulas (31)-(35).

6 Plane waves

In order to clarify the possible Feigel effect, we need to analyse not only the waves inside the sample (as was done originally in [9]) but also the waves in the outside vacuum space. The appropriate qualitative picture is as follows: Let us put the magnetoelectric matter between the two parallel planes $S_1 = \{x = -\ell\}$ and $S_2 = \{x = +\ell\}$. Then the external crossed electric and magnetic fields with the field lines parallel to these boundaries will induce the magnetoelectric covector $m = m\, dx$ along the x-axis, i.e., orthogonally to the surfaces S_1 and S_2. Outside of the matter, we have the "bath" of the virtual photons some of which will penetrate the interior of the sample, reflecting and refracting at its boundaries. Obviously, the largest contribution to the possible effect should come from the electromagnetic waves which travel along the x-axis, i.e., with the wave vectors normal to the boundaries. Clearly, for each right-moving wave, falling on the left boundary S_1, there exists an equal but opposite left-moving wave, falling on the right boundary S_2. The contributions of these *ingoing* waves to the momentum density of the electromagnetic field are equal with the opposite sign, thus providing a *balance* of the light pressures in the left and in the right vacuum regions. However, we have to find the contributions of the *outgoing* waves. If they turn out to be different in the left and in the right vacuum regions, this would seemingly yield a violation of the momentum balance and would encompass a nontrivial Feigel effect.

There are three regions: 1) the left vacuum space (for $x < -\ell$), 2) the right vacuum space (for $x > \ell$), and 3) the interior region filled with the magnetoelectric matter (for $-\ell < x < \ell$). The configurations of the electromagnetic field in these three domains read, respectively, as follows.

1) In the first region ($x < -\ell$):

$$E = (R_1 + L_1)dy, \qquad B = \frac{k}{\omega}(R_1 - L_1)dx \wedge dy, \tag{53}$$

$$\mathcal{D} = -\,\varepsilon_0(R_1 + L_1)dx \wedge dz, \qquad \mathcal{H} = \frac{k}{\mu_0\omega}(R_1 - L_1)dz. \tag{54}$$

Here $R_1 = R_1(\omega t - kx)$ and $L_1 = L_1(\omega t + kx)$ describe the right- and left-moving waves, respectively. With $k = \omega/c$ one can straightforwardly check that this configuration is a solution of the Maxwell equations.

2) In the second region ($x > \ell$):

$$E = (R_2 + L_2)dy, \qquad B = \frac{k}{\omega}(R_2 - L_2)dx \wedge dy, \tag{55}$$

$$\mathcal{D} = -\,\varepsilon_0(R_2 + L_2)dx \wedge dz, \qquad \mathcal{H} = \frac{k}{\mu_0\omega}(R_2 - L_2)dz. \tag{56}$$

Now $R_2 = R_2(\omega t - kx)$ and $L_2 = L_2(\omega t + kx)$ describe the right- and left-moving wave, respectively, and again $k = \omega/c$.

3) In the third, the interior region ($-\ell < x < \ell$):

$$E = (R_3 + L_3)dy, \qquad B = \frac{1}{\omega}(k_+R_3 - k_-L_3)dx \wedge dy, \tag{57}$$

$$\mathcal{D} = -\left[\left(\varepsilon\varepsilon_0 + \frac{mk_+}{\omega}\right)R_3 + \left(\varepsilon\varepsilon_0 - \frac{mk_-}{\omega}\right)L_3\right]dx \wedge dz, \tag{58}$$

$$\mathcal{H} = \left[\left(\frac{k_+}{\mu\mu_0\omega} - m\right)R_3 + \left(\frac{-k_-}{\mu\mu_0\omega} - m\right)L_3\right]dz. \tag{59}$$

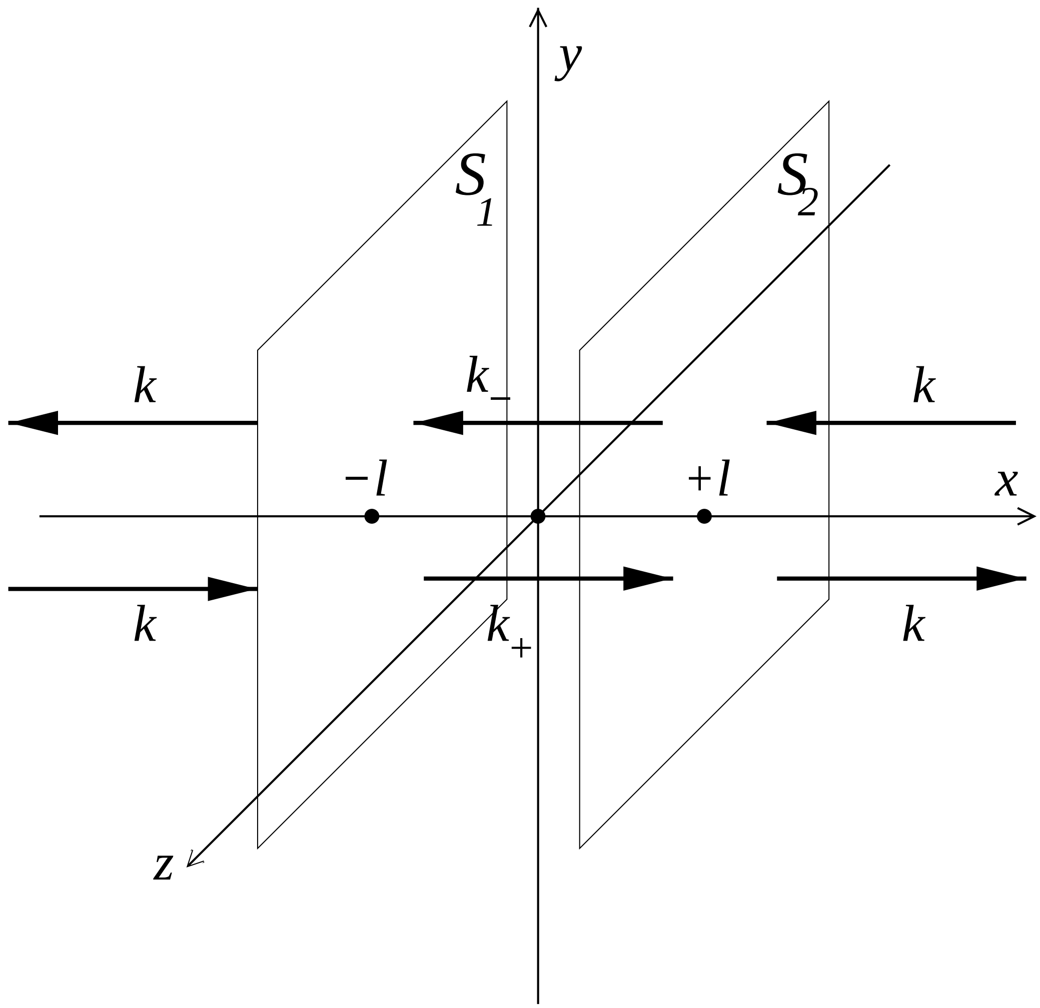

Figure 1: *Plane waves*

The matter is characterized by the electric, the magnetic, and the magnetoelectric constants ε, μ, m (the latter is determined by the external crossed fields applied to the sample). The right- and left-movers are now described by $R_3 = R_3(\omega t - k_+ x)$ and $L_3 = L_3(\omega t + k_- x)$. The birefringence, encoded in the optical metrics (42), (43), is manifest in the inequality $k_+ \neq k_-$. Explicitly, we find

$$k_\pm = \frac{n\omega}{c}\left(\sqrt{\overline{m}^2 + 1} \pm \overline{m}\right). \tag{60}$$

As before, we use here the dimensionless magnetoelectric variable $\overline{m} := m/\lambda$.

Now, we assume that the *ingoing* waves, falling on the surfaces of the sample, are *harmonic* and have equivalent structure, i.e., that

$$R_1(\omega t - kx) = a_1 \cos(\omega t - kx) + a_2 \sin(\omega t - kx), \tag{61}$$
$$L_2(\omega t + kx) = a_1 \cos(\omega t + kx) + a_2 \sin(\omega t + kx), \tag{62}$$

with the prescribed constant a_1, a_2. Then, the *outgoing* waves will also be harmonic,

$$L_1(\omega t + kx) = b_1 \cos(\omega t + kx) + b_2 \sin(\omega t + kx), \tag{63}$$
$$R_2(\omega t - kx) = c_1 \cos(\omega t - kx) + c_2 \sin(\omega t - kx), \tag{64}$$

as well as the transmitted waves inside the sample,

$$R_3(\omega t - k_+ x) = p_1 \cos(\omega t - k_+ x) + p_2 \sin(\omega t - k_+ x), \tag{65}$$
$$L_3(\omega t + k_- x) = q_1 \cos(\omega t + k_- x) + q_2 \sin(\omega t + k_- x). \tag{66}$$

The unknown coefficients $b_1, b_2, c_1, c_2, p_1, p_2, q_1, q_2$ are then uniquely determined from the jump conditions for the electromagnetic field strength and excitations at the boundaries S_1 and S_2. There are twelve jump conditions – six for every boundary surface, as usual. They read:

$$\left(\mathcal{D}_{(1)} - \mathcal{D}_{(3)}\right)\Big|_{S_1} \wedge \nu = 0, \qquad \tau_A \rfloor \left(\mathcal{H}_{(1)} - \mathcal{H}_{(3)}\right)\Big|_{S_1} = 0, \tag{67}$$
$$\left(B_{(1)} - B_{(3)}\right)\Big|_{S_1} \wedge \nu = 0, \qquad \tau_A \rfloor \left(E_{(1)} - E_{(3)}\right)\Big|_{S_1} = 0. \tag{68}$$
$$\left(\mathcal{D}_{(3)} - \mathcal{D}_{(2)}\right)\Big|_{S_2} \wedge \nu = 0, \qquad \tau_A \rfloor \left(\mathcal{H}_{(3)} - \mathcal{H}_{(2)}\right)\Big|_{S_2} = 0, \tag{69}$$
$$\left(B_{(3)} - B_{(2)}\right)\Big|_{S_2} \wedge \nu = 0, \qquad \tau_A \rfloor \left(E_{(3)} - E_{(2)}\right)\Big|_{S_2} = 0. \tag{70}$$

Here $\nu = dx$ is the 1-form density normal to the surfaces and $\tau_1 = \partial_y, \tau_2 = \partial_z$ $(A = 1, 2)$ are the two vectors tangential to the boundaries.

Substituting (53)-(59), we find that half of the jump conditions are trivially satisfied since $B \wedge \nu = 0$ and $\mathcal{D} \wedge \nu = 0$ in all the three regions. Moreover, $\tau_1 \rfloor \mathcal{H} = 0$ and $\tau_2 \rfloor E = 0$ everywhere, too. Accordingly, we are left with only four conditions which result from the continuity of $\tau_2 \rfloor \mathcal{H}$ and $\tau_1 \rfloor E$ at the two boundaries. For the harmonic waves under consideration, these conditions yield the system of algebraic equations on the coefficients $b_1, b_2, c_1, c_2, p_1, p_2, q_1, q_2$. After some algebra, we can bring this system into the form of the four matrix equations relating the 2-vectors $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, $\vec{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$, $\vec{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ to $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$:

$$T(k\ell)\vec{a} - T(-k\ell)\vec{b} - \alpha T(k_+\ell)\vec{p} + \alpha T(-k_-\ell)\vec{q} = 0, \tag{71}$$

$$T(k\ell)\vec{a} - T(-k\ell)\vec{c} + \alpha T(-k_+\ell)\vec{p} - \alpha T(k_-\ell)\vec{q} = 0, \tag{72}$$

$$-T(k\ell)\vec{a} - T(-k\ell)\vec{b} + T(k_+\ell)\vec{p} + T(-k_-\ell)\vec{q} = 0, \tag{73}$$

$$-T(k\ell)\vec{a} - T(-k\ell)\vec{c} + T(-k_+\ell)\vec{p} + T(k_-\ell)\vec{q} = 0. \tag{74}$$

Here we abbreviated

$$\alpha := \sqrt{\frac{\varepsilon}{\mu}\,(1 + \overline{m}^2)}, \tag{75}$$

and the matrix of the 2-dimensional rotation (and element of the $SO(2, R)$ group) is denoted

$$T(\varphi) := \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}. \tag{76}$$

Evidently $T(\varphi_1)T(\varphi_2) = T(\varphi_2)T(\varphi_1) = T(\varphi_1 + \varphi_2)$.

The solution of the system (71)-(74) reads:

$$\vec{p} = \frac{1}{2\Delta}\,K\left[(1+\alpha)T(k_-\ell) - (1-\alpha)T(-k_-\ell)\right]T(k\ell)\,\vec{a}, \tag{77}$$

$$\vec{q} = \frac{1}{2\Delta}\,K\left[(1+\alpha)T(k_+\ell) - (1-\alpha)T(-k_+\ell)\right]T(k\ell)\,\vec{a}, \tag{78}$$

$$\begin{aligned}\vec{b} = \frac{1}{4\Delta}\,K\big\{&(1-\alpha^2)\left[T((k_+ + k_-)\ell) - T(-(k_+ + k_-)\ell)\right] \\ &+ 4\alpha\,T((k_+ - k_-)\ell)\big\}T(2k\ell)\,\vec{a},\end{aligned} \tag{79}$$

$$\begin{aligned}\vec{c} = \frac{1}{4\Delta}\,K\Big\{&(1-\alpha^2)\left[T((k_+ + k_-)\ell) - T(-(k_+ + k_-)\ell)\right] \\ &+ 4\alpha\,T((k_- - k_+)\ell)\Big\}T(2k\ell)\,\vec{a}.\end{aligned} \tag{80}$$

Here we introduced the positive scalar function

$$\Delta := 4\alpha^2 \cos^2[(k_+ + k_-)\ell] + (1+\alpha^2)^2 \sin^2[(k_+ + k_-)\ell] \tag{81}$$

and the operator represented by the 2×2 matrix

$$K := (1+\alpha)^2 T[-(k_+ + k_-)\ell] - (1-\alpha)^2 T[(k_+ + k_-)\ell]. \tag{82}$$

Both these quantities depend on the sum

$$k_+ + k_- = \frac{2n\omega}{c}\sqrt{\overline{m}^2 + 1}. \tag{83}$$

However, since for the magnetoelectric matter the difference

$$k_+ - k_- = \frac{2n\omega}{c}\,\overline{m} \tag{84}$$

is nontrivial, the amplitudes of the *left-moving waves in matter* are clearly distinct from that of the *right-moving waves*, cf. (77), (78). The same applies to the outgoing waves in the two vacuum regions: The amplitudes of these waves are different in the first and in the second regions, cf. (79), (80). We have to check now if such a difference can yield different field momentum densities in these regions.

6.1 Energy-momentum density

In vacuum, the energy-momentum of the electromagnetic is consists of four pieces: The *energy* density 3-form

$$u := \frac{1}{2}\left(E \wedge \mathcal{D} + B \wedge \mathcal{H}\right)\,, \tag{85}$$

the *energy flux* density (or Poynting) 2-form

$$s := E \wedge \mathcal{H}\,, \tag{86}$$

the *momentum* density 3-form

$$p_a := -\,B \wedge (e_a \rfloor \mathcal{D})\,, \tag{87}$$

and the Maxwell *stress* (or momentum flux density) 2-form of the electromagnetic field

$$
\begin{aligned}
S_a \;:=\; & \frac{1}{2}\Big[(e_a \rfloor E)\wedge \mathcal{D} - (e_a\rfloor \mathcal{D})\wedge E \\
& + (e_a\rfloor \mathcal{H})\wedge B - (e_a\rfloor B)\wedge \mathcal{H}\Big]\,. \qquad (88)
\end{aligned}
$$

Accordingly, using (53), (54), we find for the first vacuum region:

$$u = \varepsilon_0\,(R_1^2 + L_1^2)\,dx\wedge dy\wedge dz, \qquad (89)$$

$$s = \frac{1}{\mu_0 c}\,(R_1^2 - L_1^2)\,dy\wedge dz, \qquad (90)$$

$$p_a = \frac{\varepsilon_0}{c}\begin{pmatrix}(R_1^2 - L_1^2)\,dx\wedge dy\wedge dz\\ 0\\ 0\end{pmatrix}, \qquad (91)$$

$$S_a = -\,\varepsilon_0\begin{pmatrix}(R_1^2 + L_1^2)\,dy\wedge dz\\ -2R_1L_1\,dz\wedge dx\\ 2R_1L_1 d\,x\wedge dy\end{pmatrix}. \qquad (92)$$

In the second vacuum region we have to replace $R_1 \to R_2$ and $L_1 \to L_2$.

Only the mean averaged (over a time period) quantities have a direct physical meaning. We find for the averaged quantities in the *first region*

$$<u> = \frac{\varepsilon_0}{2}(|a|^2 + |b|^2)\,dx\wedge dy\wedge dz, \qquad (93)$$

$$<s> = \frac{1}{2\mu_0 c}\,(|a|^2 - |b|^2)\,dy\wedge dz, \qquad (94)$$

$$<p_a> = \frac{\varepsilon_0}{2c}\begin{pmatrix}(|a|^2 - |b|^2)dx\wedge dy\wedge dz\\ 0\\ 0\end{pmatrix}, \qquad (95)$$

$$<S_a> = -\,\frac{\varepsilon_0}{2}\begin{pmatrix}(|a|^2 + |b|^2)\,dy\wedge dz\\ 0\\ 0\end{pmatrix}, \qquad (96)$$

and in the *second region*

$$< u > \; = \; \frac{\varepsilon_0}{2}(|c|^2 + |a|^2)\, dx \wedge dy \wedge dz, \tag{97}$$

$$< s > \; = \; \frac{1}{2\mu_0 c}\,(|c|^2 - |a|^2)\, dy \wedge dz, \tag{98}$$

$$< p_a > \; = \; \frac{\varepsilon_0}{2c}\begin{pmatrix} (|c|^2 - |a|^2) dx \wedge dy \wedge dz \\ 0 \\ 0 \end{pmatrix}, \tag{99}$$

$$< S_a > \; = \; -\frac{\varepsilon_0}{2}\begin{pmatrix} (|c|^2 + |a|^2)\, dy \wedge dz \\ 0 \\ 0 \end{pmatrix}. \tag{100}$$

Consequently, it remains to calculate $|b|^2$, which contains the data about the left-outgoing waves, and compare it with $|c|^2$, which contains the data about the right-outgoing waves. Using (79) and (80), we find explicitly

$$|b|^2 \; = \; |a|^2 \left[1 + \frac{4\alpha}{\Delta}(1-\alpha^2)\,\sin(k_+ + k_-)\ell\,\sin(k_+ - k_-)\ell\right], \tag{101}$$

$$|c|^2 \; = \; |a|^2 \left[1 - \frac{4\alpha}{\Delta}(1-\alpha^2)\,\sin(k_+ + k_-)\ell\,\sin(k_+ - k_-)\ell\right]. \tag{102}$$

In the computation, we used the identity $(\vec{a}^{\rm T}\, T(\varphi)\, \vec{a}) = |a|^2 \cos\varphi$, which can be straightforwardly demonstrated.

As we can see, the contributions of the *outgoing* waves to the field momentum are clearly different in the two vacuum regions. The difference reads explicitly

$$|b|^2 - |c|^2 = \frac{8|a|^2\alpha(1-\alpha^2)}{\Delta}\,\sin(k_+ + k_-)\ell\,\sin(k_+ - k_-)\ell. \tag{103}$$

When $m = 0$, in view of (84) the "bath" of virtual waves around the sample is in equilibrium since then (103) vanishes. The total momentum of the waves in both vacuum regions is obviously equal to zero. However, for magnetoelectric matter, the mentioned "bath" is still balanced in the sense that the field momentum carried by the waves in the left vacuum region is the *same* as that of the waves in the right vacuum region. Namely, substituting (101) and (102) into (95) and (99), we find that the momentum density of the electromagnetic field *in both regions* is equal

$$< p_x > = -\frac{2\varepsilon_0\alpha}{c\,\Delta}\,(1-\alpha^2)\,\sin(k_+ + k_-)\ell\,\sin(k_+ - k_-)\ell\, dx \wedge dy \wedge dz. \tag{104}$$

At the same time, the stress is different in the two regions. When we compute the corresponding force $\mathcal{F}_x = \int < S_x >$, acting on the boundary of the sample, the resulting expressions will read for the first (left) and for the second (right) surfaces, respectively:

$$\mathcal{F}_x^{\rm left} = \varepsilon_0 |a|^2 \mathcal{A} \left[1 + \frac{2\alpha}{\Delta}(1-\alpha^2)\, \sin(k_+ + k_-)\ell \, \sin(k_+ - k_-)\ell \right], \quad (105)$$

$$\mathcal{F}_x^{\rm right} = -\varepsilon_0 |a|^2 \mathcal{A} \left[1 - \frac{2\alpha}{\Delta}(1-\alpha^2)\, \sin(k_+ + k_-)\ell \, \sin(k_+ - k_-)\ell \right] \quad (106)$$

Here $\mathcal{A}$ is the area of the boundary surface (we assume the left and right surfaces to be equal). Thus, there will be a nontrivial resulting force acting on the sample in the direction of the magnetoelectric vector:

$$\mathcal{F}_x^{\rm left} + \mathcal{F}_x^{\rm right} = \frac{4\varepsilon_0 |a|^2 \mathcal{A}\alpha}{\Delta}(1-\alpha^2)\, \sin(k_+ + k_-)\ell \, \sin(k_+ - k_-)\ell. \quad (107)$$

At first sight, the results obtained, namely (104) and (107), provide a theoretical *support* for the possible *Feigel effect*. For completeness, however, it is necessary to analyse also the situation when the directions of all waves are reversed, i.e., instead of assuming equal incoming waves, we should also consider the case of equal outgoing waves. Fortunately, it is not necessary to perform a new computation. All we need is to put $k \to -k$ in (53)-(66) and then notice that in the jump equations (71)-(74) we have to change the sign of $\alpha \to -\alpha$. Then repeating the computations of the amplitudes $\vec{p}, \vec{q}, \vec{b}, \vec{c}$, we arrive again to the solution (77)-(80) with the replacements $k \to -k$ and $\alpha \to -\alpha$. As a result, the total momentum density turns out to be again (104). [It is important to note that here we *do not* have to replace $\alpha \to -\alpha$, since $(|a|^2 - |b|^2)$ is changed to $(|b|^2 - |a|^2)$ in (95), and similarly, $(|c|^2 - |a|^2)$ is changed to $(|a|^2 - |c|^2)$ in (95)]. However, the resulting force computed from the stress on the left and right boundary surfaces will have the *opposite sign* (and equal magnitude) to that of (107). Correspondingly, when we consider both contributions together, the total force will be found to be equal to zero. In other words, the magnetoelectric body will not move, despite the presence of a certain asymmetry between the left- and right-moving waves in the matter.

This conclusion is based on the evident symmetry which characterizes the "bath" of the virtual photons ("vacuum fluctuations") in the regions 1 and 2: for each left-moving virtual photon there is an equal right-moving virtual photon. But actually, a different conclusion is derived for *real* electromagnetic waves and it would be interesting to verify experimentally whether a magnetoelectric sample would have a nontrivial momentum and thus would move, being inserted between the oppositely directed but otherwise equal beams of photons.

6.2 Discussion: choice of the modes

One may ask a question: Are the modes of the fluctuating (vacuum) waves counted correctly in our analysis? In particular, isn't there a possible "overcounting" which somehow affects the final conclusion? In order to check this point, let us consider a more general setting, when the incoming and the outcoming modes are not specified from the very beginning.

The wave field configurations are given, in the three regions, by the formulas (53),(54), and (55),(56), and (57)-(59), respectively. However, we now assume, that whereas the wave components for the fluctuations in the *first region* are given by (61) and (63), and for the waves *inside matter* by (65) and (66) (as before), in the *second region* (left vacuum half-space), instead of (62) and (64) we use a more general ansatz

$$R_2(\omega t - kx) = \widetilde{a}_1 \cos(\omega t - kx) + \widetilde{a}_2 \sin(\omega t - kx), \tag{108}$$

$$L_2(\omega t + kx) = \widetilde{b}_1 \cos(\omega t + kx) + \widetilde{b}_2 \sin(\omega t + kx). \tag{109}$$

By putting $\widetilde{a}_{1,2} = c_{1,2}$ and $\widetilde{b}_{1,2} = a_{1,2}$, we recover the previous choice of the modes (62) and (64), i.e., the equal incoming waves from the both (left and right) sides.

With the ansatz (108), (109), we can choose any other possible modes. We can straightforwardly generalize our earlier computations now. In particular, by using (108), (109) in the jump conditions (67)-(70), we find instead of the system (71)-(74) the following set of algebraic conditions:

$$T(k\ell)\vec{a} - T(-k\ell)\vec{b} - \alpha T(k_+\ell)\vec{p} + \alpha T(-k_-\ell)\vec{q} = 0, \tag{110}$$

$$-T(-k\ell)\vec{\widetilde{a}} + T(k\ell)\vec{\widetilde{b}} + \alpha T(-k_+\ell)\vec{p} - \alpha T(k_-\ell)\vec{q} = 0, \tag{111}$$

$$-T(k\ell)\vec{a} - T(-k\ell)\vec{b} + T(k_+\ell)\vec{p} + T(-k_-\ell)\vec{q} = 0, \tag{112}$$

$$-T(-k\ell)\vec{\widetilde{a}} - T(k\ell)\vec{\widetilde{b}} + T(-k_+\ell)\vec{p} + T(k_-\ell)\vec{q} = 0. \tag{113}$$

The equations (110) and (112) provide the relation between $\vec{a}, \vec{b}$ and $\vec{p}, \vec{q}$, whereas the equations (111) and (113) provide the relation between $\vec{\widetilde{a}}, \vec{\widetilde{b}}$ and

$\vec{p}, \vec{q}$. By excluding the pair $\vec{p}, \vec{q}$, we eventually find the direct relation between the amplitudes in the left and in the right vacuum regions:

$$\begin{aligned}
\vec{\widetilde{a}} &= \frac{1}{4\alpha} T[(k_- - k_+)\ell]\Big\{ K T(2k\ell)\,\vec{a} \\
&\qquad + (1-\alpha^2)(T[(k_+ + k_-)\ell] - T[-(k_+ + k_-)\ell]\vec{b}\Big\}, \qquad (114) \\
\vec{\widetilde{b}} &= \frac{1}{4\alpha} T[(k_- - k_+)\ell]\Big\{ K' T(-2k\ell)\,\vec{b} \\
&\qquad + (1-\alpha^2)(T[-(k_+ + k_-)\ell] - T[(k_+ + k_-)\ell]\vec{a}\Big\}. \qquad (115)
\end{aligned}$$

Here, cf. with (82), we introduced the operator

$$K' := (1+\alpha)^2 T[(k_+ + k_-)\ell] - (1-\alpha)^2 T[-(k_+ + k_-)\ell]. \qquad (116)$$

As a check, we can easily verify that by choosing the modes as $\widetilde{a}_{1,2} = c_{1,2}$ and $\widetilde{b}_{1,2} = a_{1,2}$, the formulas (114) and (115) reduce to the old results (79) and (80).

Now, however, let us make a different choice of the modes. Namely, let us assume that $\vec{b} = \vec{a}$. In other words, we select in the "bath" of the vacuum fluctuations in the first (left) region a pair of modes with the equal intensities, one of which is a left-mover, and another is a right-mover. Clearly, we can always select in the spectrum of all fluctuations (which contains any possible modes) such pairs of the equal intensities. When we compute the contribution of these modes to the total momentum, we evidently find the zero net result, since $|a|^2 = |b|^2$, see eq. (95). Summing over *all* such pairs, we then certainly obtain that the total net momentum of the vacuum fluctuations in the left half-space is equal zero. Now, let us look to the waves which correspond to any such pair in the second (right) vacuum half-space. The corresponding amplitudes are given by (114) and (115), where we have to put $\vec{b} = \vec{a}$. It thus remains to calculate the intensities of these waves. The direct calculation yields

$$\begin{aligned}
|\widetilde{a}|^2 = |\widetilde{b}|^2 = |a|^2 \Big\{ 1 &+ \frac{1-\alpha^2}{2\alpha} \sin[2(k_+ + k_-)\ell]\sin(2k\ell) \\
&+ \frac{1-\alpha^2}{2\alpha^2} \sin^2[(k_+ + k_-)\ell]\,\big[(1-\alpha^2) - (1+\alpha^2)\cos(2k\ell)\big]\Big\}. \qquad (117)
\end{aligned}$$

Accordingly, we find that the corresponding pairs of waves in the right region *also have equal intensities* and thus they cancel each other in the expression of the net momentum! The latter is easily seen from (99), properly generalized by replacing $|c|^2 - |a|^2$ with $|\widetilde{a}|^2 - |\widetilde{b}|^2$. The sum over all pairs then yields the total trivial momentum for the vacuum waves in the second region.

In other words, our previous conclusion is completely confirmed for a different choice of the modes: The total field momentum is equal on the both sides of the sample, and thus there is no any physical reason for the sample to move in any direction.

In fact, one can check that the same conclusion remains valid *for any* other choice of the vacuum modes, in the sense that the total momentum in the left vacuum region is always balanced by the *same* total momentum in the right vacuum region.

For completeness, let us also analyse the the stress and the corresponding force which can be computed along the same lines as in the previous section. The stress in the first (left) region is again given by the formula (96), whereas the in the expression (100) for the stress in the second (right) region we have to replace $|c|^2 + |a|^2$ with $|\widetilde{a}|^2 + |\widetilde{b}|^2$. Now, a straightforward computation yields for the resulting force

$$\begin{aligned} \mathcal{F}_x^{\rm left} + \mathcal{F}_x^{\rm right} &= \frac{\varepsilon_0 |a|^2 \mathcal{A}\,(\alpha^2 - 1)}{2\alpha^2}\Big\{\alpha\,\sin[2(k_+ + k_-)\ell]\sin(2k\ell) \\ &+ \sin^2[(k_+ + k_-)\ell]\,\big[(1-\alpha^2) - (1+\alpha^2)\cos(2k\ell)\big]\Big\}. \quad (118) \end{aligned}$$

Although this, like (107) above, seem to describe a nontrivial force acting on the material sample, we have to take into account that the two wave "baths" in the left and in the right regions are actually on the equal footing. Thus, in order to treat them equally, we have to consider the symmetric situation when the *original* vacuum fluctuations with the equal amplitudes are in the second (right) region, whereas the *secondary* vacuum waves, which arise due to the refraction and transition through the medium of the original fluctuations, are in the first (left) region. No new computations are actually needed since this situation is easily obtained from the previous one by the interchange of the quantities with and without the tildes. Then, for such a symmetric situation, the momentum is again the same (and equal zero) in both regions, whereas the resulting force if given by (118) with an *opposite sign.* Consequently, when we put together both pieces of the picture, we find the total force equal zero.

One may wonder if in the above analysis we do not make an "overcounting". There are several arguments which demonstrate that the answer is negative. Firstly, when thinking about the classical electromagnetic waves which were used by Feigel (and, following him, by us) to model the "vacuum fluctuations", we have to recall that the very notion of fluctuation means that this is an uncontrollable process. In this sense, it is unreasonable to assume that the original vacuum fluctuations may occur only in the first (left) region, whereas the second region is merely an "arena" for the secondary waves. The original waves may (and should) emerge in the second (right) region and produce the secondary waves in the first region. This is precisely what we described above. Secondly, as we see from the formulas (114) and (115) which relate the original and the secondary waves, the latter always have a *different* polarization as compared to that of the former (in simple terms, the electric vector in the wave is rotated, as clearly shown by the above formulas). Since the waves with different polarizations obviously are linearly independent, we come to the conclusion that the contribution of the original waves, say, in the first region, are *not* overcounted by the contribution of the secondary waves in the same region.

7 Conclusion

Our axiomatic covariant approach to electrodynamics provides tools to check the theoretical input of the Feigel effect. In our study, we have performed the following steps needed for the re-evaluation of the possibility of the Feigel process:

- We constructed the correct constitutive relation for an arbitrarily moving medium with magnetoelectric properties.
- We derived the relativistic dynamics of such a medium.
- We directly computed forces and momenta with the help of the energy-momentum of electromagnetic field in vacuum by investigating the "bath" of the vacuum waves in the two vacuum regions outside of the magnetoelectric medium.
- We obtained the generally covariant expression for the field momentum as a part of the total energy-momentum of the physical system (matter+field).

Concerning the last point, it is worthwhile to mention that the properties of the resulting energy-momentum are compatible with all theoretical and experimental criteria [20]. In particular: 1) the energy-momentum is derived from first principles: the Lorentz force acting on the free and bound charges and currents. 2) This tensor is explicitly symmetric (without the *ad hoc* Abraham term). 3) It provides Planck's field-theoretic $\mathbf{p} = \mathbf{s}/c^2$ generalization of the relation $\Delta m = \Delta E/c^2$. 4) It is compatible with experiment, including wave phenomena in dielectrics (Jones et al [26]), the measurement the torque in crossed fields (Walker & Walker [27]), and the measurement of an axial force in crossed fields (James [28]).

On the basis of our study, we come to the following conclusions: The derivation of the generally covariant relativistic constitutive relations for a moving magnetoelectric medium, together with the subsequent analysis of the vacuum waves travelling through the sample of a finite size shows that the magnetoelectric body will *not* move, despite the presence of a certain asymmetry between the left- and right-moving waves in the matter. However, this only refers to the case of waves due to *vacuum fluctuations.*

For the *real waves* falling symmetrically from the two sides on a magnetoelectric body, we expect a *nontrivial effect of the Feigel type.* Thus, we cannot confirm the possibility of "extracting momentum from nothing". This conclusion can be further supported by the investigation of the energy-momentum of the system (matter+field) which reveals the non-relativistic expressions for the momentum and energy that are different from those used by Feigel in his work. Certainly, it is worthwhile to recall that in our study, we applied classical electrodynamics to a dielectric medium by well-established methods. This is exactly the same type of approach that was used by Feigel himself.

A deeper analysis, using quantum field theoretical methods, could be desirable. The application of the methods of quantum field theory could possibly improve the understanding of the relevant physics, in particular, might clarify the mechanism of the counting of the fluctuating modes. As concerns the nontrivial Feigel-type effect for the real (non-vacuum) electromagnetic waves, the corresponding experimental scheme can be as follows: The original beam (say, of a laser) can be split into two beams by a simple mirror system and then both beams can be directed (from the two sides) on a magnetoelectric sample, bringing the latter into a motion. The corresponding velocity can be estimated using the results of Sec. 6.1; for the thin samples the effect should be proportional to the value of the magnetoelectric parameter $\overline{m}$.

References

[1] J.A. Schouten, *Tensor Analysis for Physicists.* 2nd ed. reprinted (Dover: Mineola, New York 1989).

[2] E.J. Post, *Formal Structure of Electromagnetics – General Covariance and Electromagnetics* (North Holland: Amsterdam, 1962, and Dover: Mineola, New York, 1997).

[3] F.W. Hehl and Yu.N. Obukhov, *Foundations of Classical Electrodynamics: Charge, flux, and metric* (Birkhäuser: Boston, 2003) *Progress in Math. Physics*, vol. **33**, 430 pp.

[4] I.V. Lindell. Differential Forms in Electromagnetics. IEEE–Wiley-Interscience, New York (2004).

[5] D.H. Delphenich, *On the axioms of topological electromagnetism,* Ann. Phys. (Leipzig) **14** (2005) 347-377; updated version of arXiv.org/hep-th/0311256.

[6] H. Minkowski, *Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern*, Nachr. Ges. Wiss. Göttingen (1908) 53-111.

[7] M. Abraham, *Zur Elektrodynamik bewegter Körper*, *Rend. Circ. Mat. Palermo* **28** (1909) 1-28; M. Abraham, *Sull'elettrodinamica di Minkowski*, *Rend. Circ. Mat. Palermo* **30** (1910) 33-46.

[8] A. Einstein and J. Laub, *Über die im elektromagnetischen Felde auf ruhende Körper ausgeübten pondermotorischen Kräfte*, *Ann. Phys. (Leipzig)* **26** (1908) 541-551.

[9] A. Feigel, *Quantum vacuum contribution to the momentum of dielectric media*, *Phys. Rev. Lett.* **92** (2004) 020404 (4 pages).

[10] R. Schützhold and G. Plunien, *Comment on "Quantum vacuum contribution to the momentum of dielectric media"*, *Phys. Rev. Lett.* **93** (2004) 268901.

[11] A. Feigel, *Feigel replies*, *Phys. Rev. Lett.* **93** (2004) 268902.

[12] B.A. van Tiggelen and G.L.J.A. Rikken, *Comment on "Quantum vacuum contribution to the momentum of dielectric media"*, *Phys. Rev. Lett.* **93** (2004) 268903.

[13] A. Feigel, *Feigel replies*, *Phys. Rev. Lett.* **93** (2004) 268904.

[14] T.H. O'Dell, *The Electrodynamics of Magneto-Electric Media,* North-Holland, Amsterdam (1970).

[15] T. Roth and G.L.J.A. Rikken, *Observation of magnetoelectric Jones birefringence*, *Phys. Rev. Lett.* **85** (2000) 4478-4481; G.L.J.A. Rikken, C. Strohm, and P. Wyder, *Observation of magnetoelectric directional anisotropy*, *Phys. Rev. Lett.* **89** (2002) 133005 (4 pages).

[16] A.H. Taub, *General relativistic variational principle for perfect fluids*, *Phys. Rev.* **94** (1954) 1468-1470.

[17] B.F. Schutz, *Perfect fluids in general relativity: Velocity potentials and a variational principle*, *Phys. Rev.* **D2** (1970) 2762-2773.

[18] M. Bailyn, *Variational principle for perfect and imperfect fluids in general relativity*, *Phys. Rev.* **D22** (1980) 276-279.

[19] Yu.N. Obukhov and V.A. Korotky, *The Weyssenhoff fluid in Einstein-Cartan theory*, *Class. Quantum Grav.* **4** (1987) 1633-1657; Yu.N. Obukhov and O.B. Piskareva, *Spinning fluid in general relativity*, *Class. Quantum Grav.* **6** (1989) L15-L19.

[20] Yu.N. Obukhov and F.W. Hehl, *Electromagnetic energy-momentum and forces in matter*, *Phys. Lett.* **A311** (2003) 277-284.

[21] W. Gordon, *Zur Lichtfortpflanzung nach der Relativitätstheorie*, Ann. *Phys. (Leipzig)* **72** (1923) 421-456.

[22] P. Poincelot, *Sur l'expression de la densité d'énergie électromagnetique*, *C.R. Acad. Sci. Paris, Série B* **264** (1967) 1064-1066; P. Poincelot, *Sur le tenseur électrodynamique*, *C.R. Acad. Sci. Paris, Série B* **264** (1967) 1179-1181; P. Poincelot, *Sur la symétrie du tenseur électrodynamique*, *C.R. Acad. Sci. Paris, Série B* **264** (1967) 1225-1226; P. Poincelot, *Sur*

le tenseur d'impulsion-énergie électromagnetique, C.R. Acad. Sci. Paris, *Série B* **264** (1967) 1560-1562.

[23] H.F. Tiersten and C.F. Tsai, *On the interaction of the electromagnetic field with heat conducting deformable insulators*, J. Math. Phys. **13** (1972) 361-378.

[24] D.F. Nelson, *Momentum, pseudomomentum, and wave momentum: Toward resolving the Minkowski-Abraham controversy*, Phys. Rev. **A44** (1991) 3985-3996.

[25] R. Loudon, *Theory of the radiation pressure on dielectric surfaces*, J. Mod. Opt. **49** (2002) 821-838.

[26] R.V. Jones and J.C.S. Richards, *The pressure of radiation in a refracting medium*, Proc. Roy. Soc. Lond., Ser. Math. Phys. Sci. **221** (1954) 480-498; R.V. Jones and B. Leslie, *The measurement of optical radiation pressure in dispersive media*, Proc. Roy. Soc. Lond. **A360** (1978) 347-363.

[27] G.B. Walker, D.G. Lahoz, and G. Walker, *Measurement of the Abraham force in Barium Titanate specimen*, Can. J. Phys. **53** (1975) 2577-2586; G.B. Walker, and G. Walker, *Mechanical forces in a dielectric due to electromagnetic field*, Can. J. Phys. **55** (1977) 2121-2127.

[28] R.P. James, *Force on permeable matter in time-varying fields*, Ph.D. Thesis (Dept. of Electrical Engineering, Stanford Univ.: 1968); R.P. James, *A "simplest case" experiment resolving the Abraham-Minkowski controversy on electromagnetic momentum in matter*, Proc. Nat. Acad. Sci. **61** (1968) 1149-1150.