



On the effect of the global cosmological expansion on the local dynamics in the Solar System

Final Report

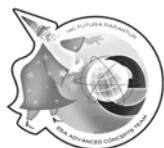
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On the effect of the global cosmological expansion on the local dynamics in the Solar System

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Abstract

Modern cosmology is described within the framework of General Relativity in which the geometry of space and time is subject to dynamical change. All current models of our Universe have in common that space is presently in a state of expansion. The question addressed in this paper is whether, and to what extent, does this expansion influence the dynamics on small scales (as compared to cosmological ones), particularly in our Solar System. Here our reference order of magnitude for any effect is given by the apparent anomalous acceleration of the Pioneer 10 and 11 spacecrafts, which is of the order of 10^{-9}m/s^2 . We distinguish between *dynamical* and *kinematical* effects and critically review the status of both as presented in the current literature. We argue that in the Solar System dynamical effects can be safely modeled by suitably improved Newtonian equations, showing that effects do exist but are smaller by many orders of magnitude compared to our Pioneer reference. On the other hand, the kinematical effects need a proper relativistic treatment and have been argued by others to give rise to an additional acceleration of the order Hc , where H is the Hubble parameter and c is the velocity of light. This simple and suggestive expression is intriguingly close to the anomalous Pioneer acceleration. We reanalyzed this argument and found a discrepancy by a factor of $(v/c)^3$, which strongly suppresses the alleged Hc -effect for the Pioneer spacecrafts by 13 orders of magnitude. We conclude with a general discussion which stresses the fundamental importance to understand precisely, i.e. within the full dynamical theory (General Relativity), the back-reaction effects of local inhomogeneities for our interpretation of cosmological data, a task which is not yet fully accomplished. Finally, a structured literature list of more than 80 references gives an overview over the relevant publications (known to us).

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1 Introduction

1.1 Aim of the study

The overall theme of this study concerns the question of whether the global cosmological expansion has any influence on the local dynamics and kinematics within the Solar System. Despite many efforts in the past, this problem is still debated controversially in the current literature. Hence there is room for speculations that such an influence might also be (partly) responsible for the apparently anomalous acceleration of the Pioneer spacecrafts, the so-called ‘Pioneer-Anomaly’ [73, 74, 77, 78, 79, 80], henceforth abbreviated by PA. Existing investigations in this direction, such as those already mentioned in ESA’s original call for proposals (CfP), arrive at partially conflicting conclusions. To resolve this issues, the CfP originally suggested a very fundamental line of attack, namely to investigating the *double embedding-problem* for the Solar System. Strictly speaking this means to find a solution to Einstein’s field equations that match 1) the gravitational field (i.e. the spacetime metric $g_{\mu\nu}$) of the Sun to that of the intergalactic neighbourhood, and 2) to match the Galactic gravitational field to that of its cosmic background.

Analytical solutions are known for some embedding problems under various special assumptions concerning symmetries and the matter content of the cosmological environment:

- A Spherical symmetry, matching Schwarzschild to a Friedmann-Robertson-Walker (FRW) universe with pressureless matter and zero cosmological constant Λ . [7, 33].
- B Obvious generalizations of A to many, non-overlapping regions— so-called ‘swiss-cheese models’.
- C Generalizations of A,B for $\Lambda \neq 0$ [1].
- D Generalizations of A to spherically symmetric but inhomogeneous Lemaître-Tolman-Bondi cosmological backgrounds [3].

In all these approaches, which are based on the original idea of Einstein & Straus [7], the matching is between a strict vacuum solution (Schwarzschild) and some cosmological model. Hence, within the matching radius, the solution is strictly of the Schwarzschild type, so that there will be absolutely no dynamical influence of cosmological expansion on local dynamical processes *by construction*. The only relevant quantity to be determined is the matching radius, r_S , (sometimes called the ‘Schücking radius’) as a function of the central mass and the time. This can be done quite straightforwardly and it turns out to be expressible by

$$\text{Vol}(r_S) \cdot \rho = M. \quad (1)$$

Here $\text{Vol}(r_S)$ is the 3-dimensional volume within a sphere of radius r_S in the cosmological background geometry and ρ is the (spatially constant but time dependent) cosmological mass-density. For example, for flat or nearly flat geometries we have

$\text{Vol}(r_S) = 4\pi r_S^3/3$. Thus the defining equation for the matching radius has the following simple interpretation: if you want to embed a point mass M into a cosmological background of mass density ρ you have to place it at the centre of an excised ball whose mass (represented by the left hand side of (1)) is just M . This is just the obvious dynamical condition that the gravitational pull the central mass M exerts on the ambient cosmological masses is just the same as that of the original homogeneous mass distribution within the ball. The deeper reason for why this argument works is that the external gravitational field of a spherically symmetric mass distribution is independent of its radial distribution, i.e. just depending on the total mass. This well known fact from Newton's theory remains true in General Relativity, as one readily sees by recalling the uniqueness of the exterior Schwarzschild solution. We may hence simply think of the cosmological mass inside a sphere of radius r_S as being squashed to a point (or black hole) at its centre, without affecting the dynamics outside the radius r_S . Note that for expanding universes the radius r_S expands with it, that is, it is co-moving with the cosmological matter.

A fundamental point of critique against that solution is that the location of the matching radius r_S is dynamically unstable. Slight perturbations to larger radii will let it increase without bound, slight perturbations to smaller radii will let it collapse. This can be proven rigorously but is also rather obvious, since r_S is defined by the equal and opposite gravitational pull of the central mass on one side and the cosmological mass on the other. Both pulls increase as one moves towards their side, so that the equilibrium position corresponds to a local maximum of the gravitational potential.

On a more phenomenological side one must criticize that the matching radius r_S is much too large for these models to adequately describe our actual astronomical environment. Let M be the mass of the Sun, then r_S turns out to be about 175 pc, which is more than a factor of 100 larger than the distance to our next star and more than factor of 50 larger than the average distance of stars in our Galaxy.

In short, we conclude from all this that we cannot expect much useful insight, as regards practically relevant dynamical effects within the Solar System, from further studies of models based on the Einstein-Straus matching idea.¹ In particular, double-matching, as originally proposed, will not lift these points of criticisms made above. Therefore: given the analytically complexity of matching solutions in GR, their likely low impact as regards the dynamical problem of interest, and the rather limited time of this FAST-STUDY, we proposed to not invest further analytical work in finding explicit double-matching solutions to Einstein's equations. Instead our programme contained the following points of study:

- 1 Discuss an improved Newtonian model including a cosmological expansion term. This we did and the results are given in Section 2 below. Our discussion complements [47] which just makes a perturbative analysis, thereby missing all orbits which are unstable under cosmological expansion (which do exist). In this respect it follows a very similar strategy as proposed in the recent paper by Price [64] (the basic idea of which goes back at least to Pachner's work[62, 63]), though we think that there are also useful differences. We also supply quantita-

¹ However, as stressed in Section 6, the matching problem certainly is important on larger scales.

tive estimates and clarify that the improved Newtonian equations of motion are written in terms of the right coordinates (non-rotating and metrically normalized). The purpose of this model is to develop a good physical intuition for the qualitative as well as quantitative features of any *dynamical* effects involved.

- 2 Based on first principles there are sometimes very general arguments put forward, allegedly showing the absence of any relevant dynamical effect. One such argument was given long ago by Dicke & Peebles [48]. When applied to an electromagnetically bound system like an atom in an expanding universe, its original form only involved the dynamical action principle plus some simple scaling argument. Since this reference is one of the most frequently cited in this field, and since the simplicity of the argument (which hardly involves any real analysis) is definitely deceptive, we give an independent treatment in Section 4 that is independent of any assumptions concerning the scaling behaviour of physical quantities other than spatial lengths and times. Our treatment also reveals that the original argument by Dicke & Peebles is insufficient to discuss leading order effects of cosmological expansion. It is therefore also ineffective in its attempt to contradict Pachner [62, 63].
- 3 Neither the improved Newtonian model nor other general *dynamical* arguments make any statement about possible *kinematical* effects, i.e. effects in connection with measurements of *spatial distances* and *time durations* in a cosmological environment whose geometry changes with time. This is an important issue since tracking a spacecraft means to map out its ‘trajectory’, which basically means to determine its simultaneous spatial distance to the observer at given observer times. But we know from General Relativity that the concepts of ‘simultaneity’ and ‘spatial distance’ are not uniquely defined. This fact needs to be taken due care of when analytical expressions for trajectories, e.g. solutions to the equations of motion in some arbitrarily chosen coordinate system, are compared with experimental findings. In those situations it is likely that different kinematical notions of simultaneity and distance are involved which need to be properly transformed into each other before being compared. For example, these transformations can result in additional acceleration terms which have been claimed in the literature to be directly relevant to the PA; see [69, 68, 72, 71, 70, 67]. We will confirm the existence of such effects in principle, but are in essential disagreement concerning their relevance in practice. We think that they have been overestimated by about 13 orders of magnitude. The details will be given in Section 5.
- 4 Finally we made a systematic scan of the literature on the subject. The papers found to be relevant are listed in the bibliography given at the end of this report.

1.2 References considered

We found that the literature contained quite a few more relevant references than originally anticipated. The problem has attracted quite some interest in recent years, both from a fundamental-theoretical and from a more practical viewpoint. It should be

clearly stated that the literature contains no theorem-like statement that is based on a *realistic* and *exact* solution to Einstein's equations and that asserts that cosmological expansion is irrelevant for local dynamics and kinematics. Besides the more applied problems concerning Solar-System satellite navigation there is hence also a strong fundamental interest in these issues that is reflected in recent works.

We subdivided the bibliography into four sections:

- 1 Papers dealing with the proper matching problem in General Relativity.
- 2 Papers dealing generally with the influence of the global cosmological expansion on local dynamics, irrespectively of whether they work within an improved Newtonian setting or in full General Relativity.
- 3 Papers discussing tentative explanations of the PA by means of gravity, mostly by referring to kinematical effects of space-time measurements in time dependent background geometries.
- 4 Measurements of the PA. These were just for our own instruction and are listed for completeness.

1.3 Scope of research

We believe one can give a fair estimation on the likely irrelevance of the *dynamical* effects in question. Given the weakness of the gravitational fields involved, estimations by Newtonian methods should give reliable figures of orders of magnitude for the motion of ordinary matter. *Kinematical* effects based on the equation for light propagation in an expanding background also turn out to be negligible, in contrast to some claims in the literature. What we will not do is to attempt finding yet more matching solutions, over and above those already discussed in the literature, for reasons discussed above. What is more important presently is to understand how possible kinematical effects can influence tracking data in actual measurements. Here the literature contains partially conflicting statements, which can be straightened out. Finally we intend to suggest fruitful lines for further research, based on the advances already made in the recent literature.

2 The Newtonian approach

In order to gain intuition we consider a simple bounded system, say an atom or a planetary system, immersed in an expanding cosmos. We ask for the effects of this expansion on our local system. Does our system expand with the cosmos? Does it expand only partially? Or does it not expand at all?

2.1 The two-body problem in an expanding universe

Take a two-body problem with a $1/r^2$ attractive force between them. For simplicity we think of one mass as being much smaller than the other one (this is inessential). This

can e.g. be a system consisting of two galaxies, a star and a planet (or spacecraft), or a (classical) atom given by an electron orbiting around a proton. We think this system as being immersed in an expanding universe and we model the effect of the cosmological expansion by adding to the attraction term an extra term coming from the Hubble law $\dot{r} = Hr$. Here $H := \dot{a}/a$ is the Hubble parameter, $a(t)$ the cosmological scale factor, and r the distance – as measured in the surface of constant cosmological time t – of two objects that follow the Hubble flow (cosmological expansion). The acceleration that results from the Hubble law is

$$\ddot{r}|_{\text{cosm. acc.}} = \frac{\ddot{a}}{a}r. \quad (2)$$

Note that, in the sense of General Relativity, a body that is co-moving with the cosmological expansion is moving on an inertial trajectory, i.e. it moves force free. Forces in the Newtonian sense are now the cause for *deviations* from the co-moving acceleration described by (2). This suggests that in Newton's law, $m\ddot{\vec{r}} = \vec{F}$, we have to replace $\ddot{\vec{r}}$ by $\ddot{\vec{r}} - (\ddot{a}/a)\vec{r}$. This can be justified rigorously by using the equation of geodesic deviation in General Relativity. In order to do this one must make sure that the Newtonian equations of motion are written in appropriate coordinates. That is, they must refer to a (locally) non-rotating frame and directly give the spatial geodesic distance. This is achieved by using Fermi normal coordinates along the worldline of a geodesically moving observer—in our case e.g. the Sun or the proton—, as correctly emphasized in [47]. The equation of geodesic deviation in these coordinates now gives the variation of the spatial geodesic distance to a neighbouring geodesically moving object—in our case e.g. the planet (or spacecraft) or electron. It reads²

$$\frac{d^2x^k}{d\tau^2} + R^k{}_{0l0}x^l = 0. \quad (3)$$

Here the x^k are the spatial non-rotating normal coordinates whose values directly refer to the proper spatial distance. In these coordinates we further have [47]

$$R^k{}_{0l0} = -\delta_l^k \ddot{a}/a \quad (4)$$

on the worldline of the first observer, where the overdot refers to differentiation with respect to the cosmological time, which reduces to the eigentime along the observer's worldline.

Neglecting large velocity effects (i.e. terms quadratic or higher order in v/c) we can now write down the equation of motion for the familiar two-body problem. After specification of a scale function $a(t)$, we get two ODEs for the variables (r, φ) , which describe the position³ of the orbiting body with respect to the central one:

$$\ddot{r} = \frac{L^2}{r^3} - \frac{C}{r^2} + \frac{\ddot{a}}{a}r \quad (5a)$$

$$r^2\dot{\varphi} = L. \quad (5b)$$

² By construction of the coordinates, the Christoffel symbols $\Gamma_{\alpha\beta}^\mu$ vanish along the worldline of the first observer. Since this worldline is geodesic, Fermi-Walker transportation just reduces to parallel transportation. This gives a non-rotating reference frame that can be physically realized by gyros taken along the worldline.

³ Recall that 'position' refers to Fermi normal coordinates, i.e. r is the radial geodesic distance to the observer at $r = 0$.

These are the (\ddot{a}/a) -improved Newtonian equations of motion for the two-body problem, where L represents the (conserved) angular momentum of the planet (or electron) per unit mass and C the strength of the attractive force. In the gravitational case $C = GM$, where M is the mass of the central body, and in the electromagnetic case $C = |Qq|/4\pi\epsilon_0 m$ (SI-unit), where Qq is the product of the two charges, m is the electron mass, and ϵ_0 the vacuum permittivity. In Sections 3 and 4 we will show how to obtain (5) in appropriate limits from the full general relativistic treatments.

We now wish to study the effect the \ddot{a} term has on the unperturbed Kepler orbits. We first make the obvious remark that this term results from the *acceleration* and not just the expansion of the universe.

We first remark that, in the concrete physical cases of interest, the time dependence of this term is negligible to a very good approximation. Indeed, putting $f := \ddot{a}/a$, the relative time variation of the coefficient of r in (2) is \dot{f}/f . For an exponential scale function $a(t) \propto \exp(\lambda t)$ (Λ -dominated universe) this vanishes, and for a power law $a(t) \propto t^\lambda$ (for example matter-, or radiation-dominated universes) this is $-2H/\lambda$, and hence of the order of the inverse age of the universe. If we consider a planet in the Solar System, the relevant time scale of the problem is the period of its orbit around the Sun. The relative error in the disturbance, when treating the factor \ddot{a}/a as constant during an orbit, is hence smaller than 10^{-9} for the planets in the Solar System. For atoms it is much smaller, of course. Henceforth we shall neglect this time-dependence of (2).

Keeping this in mind we set from now on $\ddot{a}/a = \text{const} =: A$. Taking the actual value one can write $A = -q_0 H_0^2$, where the index zero means ‘today’ and q stands for the cosmological deceleration parameter (defined by $q := -\ddot{a}/(H^2 a)$). Since the force is time independent, we can immediately integrate (5a) and get

$$\frac{1}{2}\dot{r}^2 + V(r) = E, \quad (6)$$

where the effective potential is

$$V(r) = \frac{L^2}{2r^2} - \frac{C}{r} - \frac{A}{2}r^2. \quad (7)$$

2.2 Specifying the initial-value problem

For (6) and (5b) we have to specify initial conditions $(r, \dot{r}, \varphi, \dot{\varphi})(t_0) = (r_0, v_0, \varphi_0, \omega_0)$ at the initial time t_0 . To study the solutions of the above equation for r one has to look at the effective potential. For this purpose it is very convenient to introduce a length scale and a time scale that naturally arise in the problem. The length scale is defined as the radius at which the acceleration due to the cosmological expansion has the same magnitude as the gravitational (or electromagnetic) attraction. This happens precisely at the critical radius

$$r_{\text{crit}} := \left(\frac{C}{|A|} \right)^{1/3}. \quad (8)$$

For $r < r_{\text{crit}}$ the gravitational (or electromagnetic) attraction dominates, whereas for $r > r_{\text{crit}}$ the effect of the cosmological expansion is the dominant one.

Intermezzo: Expressing the critical radius in terms of cosmological parameters

We briefly wish to point out how to express the critical radius in terms of the cosmological parameters. For this we write:

$$r_{\text{crit}} = \left(\frac{GM}{|q_0|H_0^2} \right)^{1/3} \approx \left(\frac{M}{M_\odot} \right)^{1/3} 120 \text{ pc}, \quad (9)$$

where we have used the current values $q_0 = -1/2$ and $h_0 = 0.7$. It is interesting to note that in the case of zero cosmological constant and pressureless matter we recover the Schücking gluing radius [33] (compare (1)):

$$r_S = \left(\frac{M}{(4/3)\pi\rho_m} \right)^{1/3}. \quad (10)$$

To see this, just recall that for a pressureless matter we have $q_0 = (1/2)\Omega_m - \Omega_\Lambda$. Then, for a vanishing cosmological constant, and using the definition $\Omega_m = \rho_m \cdot 8\pi G/(3H_0^2)$, one gets the above equations immediately.

In the electromagnetic case, for a proton-electron system,

$$r_{\text{crit}} = \left(\frac{|Qq|}{4\pi\epsilon_0 m |q_0| H_0^2} \right)^{1/3} \approx 30 \text{ AU}, \quad (11)$$

which is about as big as the Neptune orbit!

Back to the initial-value problem

The time scale we define is the period with respect to the unperturbed Kepler orbit (a solution to the above problem for $A = 0$) of semi-major axis r_0 . By Kepler's third law it is given by

$$T_K := 2\pi \left(\frac{r_0^3}{C} \right)^{1/2}. \quad (12)$$

It is convenient to introduce two dimensionless parameters which essentially encode the initial conditions (r_0, ω_0) .

$$\lambda := \left(\frac{\omega_0}{2\pi/T_K} \right)^2 = \frac{L^2}{Cr_0}, \quad (13)$$

$$\alpha := \text{sign}(A) \left(\frac{r_0}{r_{\text{crit}}} \right)^3 = A \frac{r_0^3}{C}. \quad (14)$$

For close to Keplerian orbits λ is close to one. For reasonably sized orbits α is close to zero. For example, in the Solar System, where $r_0 < 100 \text{ AU}$, one has $|\alpha| < 10^{-16}$. For an atom whose radius is smaller than 10^4 Bohr-radii we have $|\alpha| < 10^{-57}$.

Defining

$$x(t) := r(t)/r_0, \quad (15)$$

equations (6) and (5b) can now be written as

$$\frac{1}{2}\dot{x}^2 + (2\pi/T_K)^2 v_{\lambda,\alpha}(x) = e \quad (16)$$

$$x^2\dot{\varphi} = \omega_0, \quad (17)$$

where $e := E/r_0^2$ now plays the rôle of the energy-constant and where the reduced 2-parameter effective potential $v_{\lambda,\alpha}$ is given by

$$v_{\lambda,\alpha}(x) := \frac{\lambda}{2x^2} - \frac{1}{x} - \frac{\alpha}{2}x^2. \quad (18)$$

The initial conditions now read

$$(x, \dot{x}, \varphi, \dot{\varphi})(t_0) = (1, v_0/r_0, \varphi_0, \omega_0). \quad (19)$$

The point of introducing the dimensionless variables is that the three initial parameters (L, C, A) of the effective potential could be reduced to two: λ and α). This will be convenient in the discussion of the potential.

2.3 Discussion of the reduced effective potential

Circular orbits correspond to extrema of the effective potential (7). Expressed in terms of the dimensionless variables this is equivalent to $v'_{\lambda,\alpha}(1) = -\lambda + 1 - \alpha = 0$. By its very definition (13), λ is always nonnegative, implying $\alpha \leq 1$. For negative α (decelerating case) this is always satisfied. On the contrary, for positive α (accelerating case), this implies the existence of a critical radius

$$r_0 \leq r_{\text{crit}} \quad (20)$$

beyond which no circular orbit exists.

These orbits are stable if the considered extremum is a true minimum, i.e. if the second derivative of the potential evaluated at the critical value is positive. Now, $v''_{\lambda,\alpha}(1) = 3\lambda - 2 - \alpha = 1 - 4\alpha$, showing stability for $\alpha < 1/4$ and instability for $\alpha \geq 1/4$.

Expressing this in physical quantities we can summarize the situation as follows: in the decelerating case (i.e. for negative α or, equivalently, for negative A) stable circular orbits exist for every radius r_0 ; one just has to increase the angular velocity according to (21). On the contrary, in the accelerating case (i.e. for positive α , or, equivalently, for positive A), we have three regions:

- $r_0 < r_{\text{ub}} := (1/4)^{1/3} r_{\text{crit}} \approx 0.63 r_{\text{crit}}$, where circular orbits exist and are stable.⁴
- $r_{\text{ub}} \leq r_0 \leq r_{\text{crit}}$, where circular orbits exist but are unstable.
- $r_0 > r_{\text{crit}}$, where no circular orbits exist.

⁴ 'ub' stands for 'upper bound for stable circular orbits'

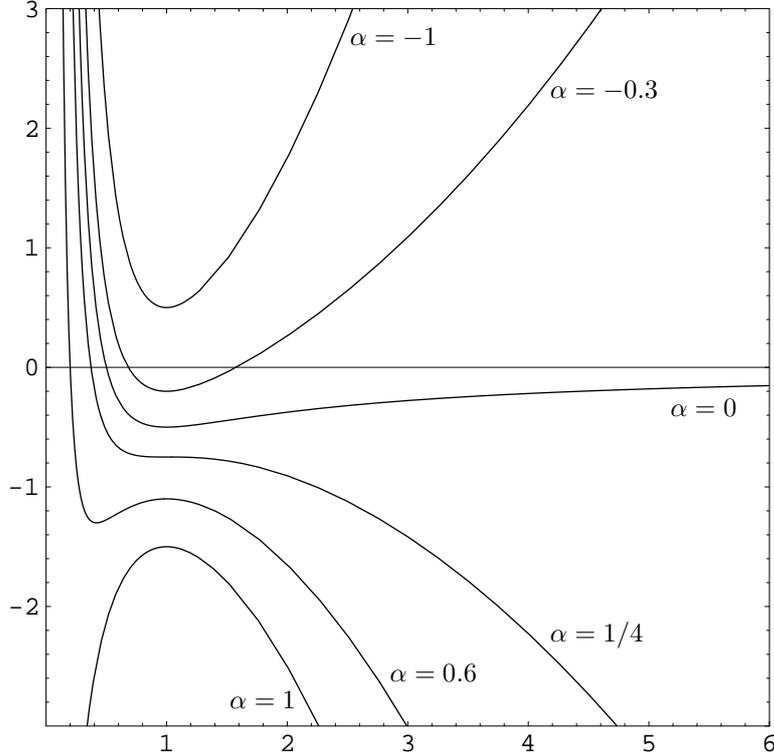


Figure 1: The figure shows the effective potential $v_{\lambda, \alpha}$, for circular orbits, where $\lambda = 1 - \alpha$, for some values of α . The initial conditions are $x = 1$ and $\dot{x} = 0$ (see (15)). At $x = 1$ the potential has an extremum, which for $\alpha < 1/4$ is a local minimum corresponding to stable circular orbits. For $1/4 \leq \alpha < 1$ these become unstable.

Generally, there exist no bounded orbits that extend beyond the critical radius r_{crit} , the reason being simply that there is no $r > r_{\text{crit}}$ where $V'(r) > 0$. Bigger systems will just be slowly pulled apart by the cosmological acceleration and approximately move with the Hubble flow at later times.⁵

Turning back to the case of circular orbits, we now express the condition for an extrema derived above, $\lambda = 1 - \alpha$, in terms of the physical quantities, which leads to

$$\omega_0 = (2\pi/T_K) \sqrt{1 - \text{sign}(A)(r_0/r_{\text{crit}})^3}. \quad (21)$$

This equation says that, in order to get a circular orbit, our planet, or electron, must have a smaller or bigger angular velocity according to the universe expanding in an accelerating or decelerating fashion respectively. This is just what one would expect, since the effect of a cosmological ‘pulling apart’ or ‘pushing together’ must be compensated by a smaller or larger centrifugal forces respectively, as compared to the Keplerian case. Equation (21) represents a modification of the third Kepler law due to the cosmological expansion. In principle this is measurable, but it is an effect of order $(r_0/r_{\text{crit}})^3$ and hence very small indeed; e.g. smaller than 10^{-17} for a planet in the Solar System.

⁵ This genuine non-perturbative behaviour was not seen in the perturbation analysis performed in [47].

Instead of adjusting the initial angular velocity as in (21), we can ask how one has to modify r_0 in order to get a circular orbit with the angular velocity $\omega_0 = 2\pi/T_K$. This is equivalent to searching the minimum of the effective potential (18) for $\lambda = 1$. This condition leads to the fourth order equation $\alpha x^4 - x + 1 = 0$ with respect to x . Its solutions can be exactly written down using Ferrari's formula, though this is not illuminating. For our purposes it is more convenient to solve it approximatively, treating α as a small perturbation. Inserting the ansatz $x_{\min} = c_0 + c_1\alpha + O(\alpha^2)$ we get $c_0 = c_1 = 1$. This is really a minimum since $v''_{1,\alpha}(x_{\min}) = 1 + O(\alpha) > 0$. Hence we have

$$r_{\min} = r_0 \left(1 + \text{sign}(A) \left(\frac{r_0}{r_{\text{crit}}} \right)^3 + O\left((r_0/r_{\text{crit}})^6 \right) \right) \quad (22)$$

This tells us that in the accelerating (decelerating) case the radii of the circular orbits with $\omega_0 = 2\pi/T_K$ becomes bigger (smaller), again according to expectation. As an example, the deviation in the radius for an hypothetical spacecraft orbiting around the Sun at 100 AU would be just of the order of 1 mm. Since it grows with the fourth power of the distance, the deviation at 1000 AU would be of the order of 10 meters.

3 Fully relativistic treatment for gravitationally-bounded systems: The McVittie model

In physics we are hardly ever in the position to mathematically rigorously model physically realistic scenarios. Usually we are at best either able to provide approximate solutions for realistic models or exact solutions for approximate models, and in most cases approximations are made on both sides. The art of physics then precisely consists in finding the right mixture in each given case. However, in this process our intuition usually strongly rests on the existence of at least some 'nearby' exact solutions. Accordingly, in this section we seek to find exact solutions in General Relativity that, with some degree of physical approximation, model a spherically symmetric body immersed in an expanding universe.

There are basically two ways to proceed, which could be described as the 'gluing'- and the 'melting'-way respectively. In the first and simpler approach one constructs a new solution to Einstein's equations by suitably gluing together two known solutions, one corresponding to the star (i.e. Schwarzschild, if the star has negligible angular momentum), the other to a homogeneous universe (i.e. FRW). The resulting spacetime is then divided into two distinct regions, whose interiors are locally isometric to the original solutions. The Einstein-Straus-Schücking vacuole [7, 33] clearly belongs to this class, as well as its generalizations [1, 3]. The advantage of this gluing-approach is its relative analytic simplicity, since the solutions to be matched are already known. Einstein's equations merely reduce to the junction conditions along their common seam. Its disadvantage is that this gluing only works under those very special conditions which allow the glued solutions to locally persist *exactly*, and these conditions are likely to be physically unrealistic.

In the second approach one considers genuinely new solutions of Einstein's equations which only approximately resemble the spacetimes of an isolated star or a ho-

mogeneous universe at small and large spatial distances respectively. This ‘melting-together’ is the more flexible approach which therefore allows to model physically more realistic situations. Needless to say that it also tends to be analytically more complicated and that a physical interpretation is often not at all obvious. The solutions of McVittie [58] and also that of Gautreau [14] fall under this class.

Our goal is to accurately model the Solar System in the currently expanding spatially-flat universe. We already argued in Section 1 that the Einstein-Straus-Schücking vacuole model unfortunately does not apply to the Solar System since the matching radius would be much too big (see also Appendix A.2). The model of Gautreau [14] is much harder to judge. Its physical and mathematical assumptions are rather implicit and not easy to interpret as regards their suitability for the problem at hand. For example, it assumes the cosmic matter to move geodesically outside the central body, but at the same time also assumes an equation of state in the form $p(\rho)$. For $p \neq 0$ this seems contradictory in a genuinely non-homogeneous situation since pressure gradients will necessarily result in deviations from geodesic motions (see [37]). Assuming $p = 0$ (which implies a motion of cosmic matter with non-vanishing shear) Gautreau finds in [14] that orbits will spiral into the central mass simply because there is a net influx of cosmic matter and hence an attracting source of increasing strength. This is not really the kind of effect we are interested in here.

Among the models discussed in the literature the one that is best understood as regards its analytical structure as well as its physical assumptions is that of McVittie [58] for $k = 0$. It is therefore, in our opinion, the natural candidate to consider first when modeling systems like the Solar System in our expanding Universe. As emphasized above, this is not to say that this model is to be considered realistic in all its detailed aspects, but at least we do have some fairly good control over the assumptions it is based on thanks to the carefully analysis by Nolan [29, 30, 31]. For example, there is a somewhat unrealistic behaviour of the McVittie spacetime and the matter in it near the singularity at $r = m/2$, as discussed below. But at larger radial distances the ‘flat’ (i.e. $k = 0$) McVittie model may well give a useful description of the exterior region of a central object in an expanding spatially-flat universe, at least in the region where the radius is much larger (in geometric units) than the central mass (to be defined below). For planetary motion and spacecraft navigation in the Solar System this is certainly the case, since the ratio of the central mass to the orbital radius is of the order of $1.5 \text{ km}/1\text{AU} = 10^{-8}$.

In this section we briefly review the McVittie spacetime and look at the geodesic equation in it, showing that it reduces to (5) in an appropriate weak-field and slow-motion limit. This provides another more solid justification for the Newtonian approach we carried out in Section 2.

The flat McVittie model (from now on to be simply referred to as *the* McVittie model) is characterized by two inputs: First, one makes the following ansatz for the metric which represents an obvious attempt to melt together the Schwarzschild metric (in spatially isotropic coordinates) with the spatially flat FRW metric (37):

$$g = \left(\frac{1 - m(t)/2r}{1 + m(t)/2r} \right)^2 dt^2 - \left(1 + \frac{m(t)}{2r} \right)^4 a^2(t) (dr^2 + r^2 d\Omega^2). \quad (23)$$

Here $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ and the two time-dependent functions m and a are to be determined. It is rotationally symmetric with the spheres of constant radius being the orbits of the rotation group. Second, it is assumed that the ideal fluid (with isotropic pressure) representing the cosmological matter moves along integral curves of the vector field $\partial/\partial t$. Note that this vector field is *not* geodesic (unlike in the Gautreau model). Moreover, this model contains the implicit assumption that the fluid motion is shearless (cf. Chapter 16 of [35]).

Einstein's equations together with the equation of state determine the four functions $m(t)$, $a(t)$, $\rho(t, r)$, and $p(t, r)$. The former are equivalent to

$$(am)' = 0, \quad (24a)$$

$$8\pi\rho = 3\left(\frac{\dot{a}}{a}\right)^2, \quad (24b)$$

$$8\pi p = -3\left(\frac{\dot{a}}{a}\right)^2 - 2\left(\frac{\dot{a}}{a}\right)' \left(\frac{1+m/2r}{1-m/2r}\right). \quad (24c)$$

Here we already used the first one in order to express the derivatives of m in terms of a and its derivatives. The first equation can be immediately integrated:

$$m(t) = \frac{m_0}{a(t)}, \quad (25)$$

where m_0 is an integration constant. Below we will show that this integration constant is to be interpreted as the mass of the central particle. We will call the metric (23) together with condition (25) the *McVittie metric*.

We note that the equation of state must be necessarily space dependent. This follows directly from the equations (24b) and (24c), which imply that the density depends on the time coordinate only whereas the pressure depends on both time and space coordinates. Formally the system (24) can be looked upon in two ways: either one prescribes an equation of state and deduces from (24b), (24c), and (25) a second-order differential equation for the scale factor a , or one specifies $a(t)$ and deduces the matter density, the pressure, and hence the equation of state.

As special cases of (24) we remark that if either a or m are time independent (24a) implies that both must be time independent. This, in turn, implies that density and pressure vanish everywhere, resulting in the Schwarzschild solution in spatially isotropic coordinates. If we choose the equation of state to be that of pressureless dust, i.e. $p = 0$, we get either the Schwarzschild solution or the dust-filled FRW universe. This follows from (24c), where we must distinguish between two cases: \dot{a}/a can only be constant if it is zero, hence resulting in the Schwarzschild solution. If \dot{a}/a is not constant, equation (24c) (with (25)) implies, after a partial differentiation with respect to r , that $m_0 = 0$. This gives the homogeneous and isotropic dust-filled FRW universe. Another possible equation of state is that of a cosmological-constant. This choice implies constancy of \dot{a}/a and hence that the second term on the left hand side of (24c) vanishes. In this way one recovers the Schwarzschild-de Sitter metric in spatially isotropic coordinates.

Finally we also mention some critical aspects of the McVittie model. In fact, unless \dot{a}/a is constant, it has a singularity at $r = m/2$ where the pressure as well as some

curvature invariants diverge. The former can be immediately seen from (24c). This is clearly a result of the assumption that the fluid moves along the integral curves of $\partial/\partial t$, which become lightlike in the limit as r tends to $m/2$. Their acceleration is given by the gradient of the pressure, which diverges in that limit. For a study of the singularity at $r = m/2$ see [30, 31].

A related question concerns the global behaviour of the McVittie metric. Each hypersurface of constant time t is a complete Riemannian manifold which besides the rotational symmetry admits a discrete isometry, given in (r, θ, φ) coordinates by

$$\phi(r, \theta, \varphi) = ([m_0/2a(t)]^2 r^{-1}, \theta, \varphi). \quad (26)$$

It corresponds to a reflection at the 2-sphere $r = (m_0/2a(t))$ and shows that the hypersurfaces of constant t can be thought of as two isometric asymptotically-flat pieces joined together at the totally geodesic (being a fixed-point set of an isometry) 2-sphere $r = m_0/2a(t)$, which is minimal. Except for the time-dependent factor $a(t)$, this is just like for the slices of constant t in the Schwarzschild metric (the difference being that (26) does not extend to an isometry of the spacetime metric unless $\dot{a} = 0$). This means that the McVittie metric cannot literally be interpreted as corresponding to a point particle sitting at $r = 0$ ($r = 0$ is in infinite metric distance) in a flat FRW universe, just like the Schwarzschild metric does not correspond to a point particle sitting at $r = 0$ in Minkowski space. Unfortunately, McVittie seems to have interpreted his solution in this fashion [58] which even until recently gave rise to some confusion in the literature (e.g. [14, 36, 12]). A clarification was given by Nolan [30]. Another important issue is whether the cosmological matter satisfies some energy condition. For a discussion about this topic we refer to [29, 30].

3.1 Interpretation of the McVittie metric

We shall now present some arguments which justify calling McVittie's metric a model for a localized mass immersed in a flat FRW background. Here we basically follow [29]. As is well known, it is generally not possible in General Relativity to assign a definite mass (or energy) to a local bounded region of space (quasi-local mass). Physically sensible definitions of such a concept of quasi-local mass exist only in favourable and special circumstances, one of them being spherical symmetry. In this case the so-called Misner-Sharp energy is often employed (e.g. [28, 5, 6, 4, 17, 18]). It allows to assign an energy content to the interior region of any two-sphere of symmetry (i.e. an orbit of $SO(3)$). For the McVittie metric ((23) with (25)) the Misner-Sharp energy takes the simple and intuitively appealing form:

$$E_{\text{MS}}(g; R, t) = \frac{4}{3}\pi R^3 \rho(t) + m_0 \quad (27)$$

where henceforth we denote by R the 'areal radius' defined by (28). This shows that the total energy is given by the sum of the cosmological matter contribution and the central mass, where the mass of the central object is given by m_0 .

Another useful definition of quasi-local mass is that of Hawking [16]. According to this definition the energy contained in the region enclosed by a spatial two-sphere S

is given by a surface integral over S , whose integrand is essentially the sum of certain distinguished components of the Ricci and Weyl tensors, representing the contributions of matter and the gravitational field respectively. Applied to the McVittie metric the latter takes the value m_0 for any 2-sphere outside of and enclosing $R = 2m_0$.

3.2 Motion of a test particle in the McVittie spacetime

We are interested in the motion of a test particle (idealizing a planet or a spacecraft) in McVittie's spacetime. In [58] McVittie concluded within a slow-motion and weak-field approximation that Keplerian orbits do not expand as measured with the 'cosmological geodesic radius' $r_* = a(t)r$. Later Pachner [62] and Noerdlinger & Petrosian [61] argued for the presence of the acceleration term (2) proportional to \ddot{a}/a within this approximation scheme, hence arriving at (5a). In the following we shall show how to arrive at (5a) from the exact geodesic equation of the McVittie metric by making clear the approximations involved. In order to compare our calculation with similar ones in the recent literature (i.e. [42, 40])⁶ we will work with the so-called 'areal radius'. It corresponds to a function that can be geometrically characterized on any spherically symmetric spacetime by taking the square root of the area of the $SO(3)$ -orbit through the considered point divided by 4π . Hence it is the same as the square root of the modulus of the coefficient of the angular part of the metric. For the McVittie metric this reads

$$R(t, r) = \left(1 + \frac{m_0}{2a(t)r}\right)^2 a(t) r. \quad (28)$$

Note that for fixed t the map $r \mapsto R(t, r)$ is 2-to-1 and that $R \geq 2m_0$, where $R = 2m_0$ corresponds to $r = m_0/2a$. Hence we restrict the coordinate transformation (28) to the region $r > m_0/2a$ where it becomes a diffeomorphism onto the region $R > 2m_0$.

Reintroducing factors of c , McVittie's metric assumes the (non-diagonal) form in the region $R > 2m_0$ (i.e. $r > m_0/2a(t)$)

$$g = \left(f(R) - \left(\frac{H(t)R}{c}\right)^2\right) c^2 dt^2 + \frac{2(H(t)R/c)}{\sqrt{f(R)}} c dt dR - \frac{dR^2}{f(R)} - R^2 d\Omega^2, \quad (29)$$

where we put

$$f(R) := 1 - \frac{2m_0}{R}, \quad (30)$$

$$H(t) := \frac{\dot{a}}{a}(t). \quad (31)$$

The region $R < 2m_0$ was investigated in [31].

The equations for a timelike geodesic (i.e. parameterized with respect to eigentime) $\tau \mapsto z^\mu(\tau)$ with $g(\dot{z}, \dot{z}) = c^2$ follows via variational principle from the Lagrangian $\mathcal{L}(z, \dot{z}) = (1/2)g_{\mu\nu}(z)\dot{z}^\mu\dot{z}^\nu$. Spherical symmetry implies conservation of angular

⁶ The paper [42] contains a derivation of the effect of cosmological expansion on the periastron precession and eccentricity change in the case where the Hubble parameter $H := \dot{a}/a$ is constant.

momentum. Hence we may choose the particle orbit to lie in the equatorial plane $\theta = \pi/2$. The constant modulus of angular momentum is

$$R^2 \dot{\phi} = L. \quad (32)$$

The remaining two equations are then coupled second-order ODEs for $t(\tau)$ and $R(\tau)$. However, we may replace the first one by its first integral that results from $g(\dot{z}, \dot{z}) = c^2$:

$$\left(f(R) - \left(\frac{H(t)R}{c} \right)^2 \right) c^2 \dot{t}^2 + \frac{2(H(t)R/c)}{\sqrt{f(R)}} c \dot{t} \dot{R} - \frac{\dot{R}^2}{f(R)} - (L/R)^2 = c^2. \quad (33)$$

The remaining radial equation is given by

$$\ddot{R} - \left(f(R) - \left(\frac{H(t)R}{c} \right)^2 \right) \frac{L^2}{R^3} \quad (34a)$$

$$+ \frac{m_0 c^2}{R^2} f(R) \dot{t}^2 \quad (34b)$$

$$- R \left(\dot{H}(t) f(R)^{\frac{1}{2}} + H(t)^2 \left(1 - \frac{m_0}{R} - \left(\frac{H(t)R}{c} \right)^2 \right) \right) \dot{t}^2 \quad (34c)$$

$$- \left(\frac{m_0}{R} - \left(\frac{H(t)R}{c} \right)^2 \right) f(R)^{-1} \frac{\dot{R}^2}{R} \quad (34d)$$

$$+ 2 \left(\frac{m_0}{R} - \left(\frac{H(t)R}{c} \right)^2 \right) f(R)^{-\frac{1}{2}} c H(t) (\dot{R}/c) \dot{t} = 0, \quad (34e)$$

Recall that $m_0 = GM/c^2$, where M is the mass of the central star (the Sun in our case) in standard units (kg).

Equations (33,34) are exact. We are interested in orbits of slow-motion (compared with the speed of light) in the region where

$$2m_0 =: R_S \ll R \ll R_H := c/H. \quad (35)$$

The latter condition clearly covers all situations of practical applicability in the solar system, since the Schwarzschild radius R_S of the Sun is about $3 \text{ km} = 2 \cdot 10^{-8} \text{ AU}$ and the ‘Hubble radius’ R_H is about $13.7 \cdot 10^9 \text{ ly} = 8.7 \cdot 10^{14} \text{ AU}$.

The approximation now consists in considering small perturbations of Keplerian orbits. Let T be a typical timescale of the problem, like the period for closed orbits or else R/v with v a typical velocity. The expansion is then with respect to the following two parameters:

$$\varepsilon_1 \approx \frac{v}{c} \approx \left(\frac{m_0}{R} \right)^{\frac{1}{2}} \quad (\text{slow-motion and weak-field}), \quad (36a)$$

$$\varepsilon_2 \approx HT \quad (\text{small ratio of characteristic-time to world-age}), \quad (36b)$$

In order to make the expression to be approximated dimensionless we multiply (33) by $1/c^2$ and (34) by T^2/R . Then we expand the right hand sides in powers of the

parameters (36), using the fact that $(HR/c) \approx \varepsilon_1 \varepsilon_2$. From this and (32) we obtain (5) if we keep only terms to zero-order in ε_1 and leading (i.e. quadratic) order in ε_2 , where we also re-express R as function of t . Note that in this approximation the areal radius R is equal to the spatial geodesic distance on the $t = \text{const.}$ hypersurfaces.

4 Fully relativistic treatment for electromagnetically-bounded systems and the argument of Dicke and Peebles

In this section we show how to arrive at (5) from a fully relativistic treatment of an electromagnetically bounded two-body problem in an expanding (spatially flat) universe. This implies solving Maxwell's equations in the cosmological background (37) for an electric point charge (the proton) and then integrate the Lorentz equations for the motion of a particle (electron) in a bound orbit (cf. [44]). Equation (5) then appears in an appropriate slow-motion limit. However, in order to relate this straightforward method to a famous argument of Dicke & Peebles, we shall proceed by taking a slight detour which makes use of the conformal properties of Maxwell's equations.

4.1 The argument of Dicke and Peebles

In reference [48] Dicke & Peebles presented an apparently very general and elegant argument, that purports to show the insignificance of any dynamical effect of cosmological expansion on a local system that is either bound by electromagnetic or gravitational forces which should hold true *at any scale*. Their argument involves a rescaling of spacetime coordinates, $(t, \vec{x}) \mapsto (\lambda t, \lambda \vec{x})$ and certain assumptions on how other physical quantities, most prominently mass, behave under such scaling transformations. For example, they assume mass to transform like $m \mapsto \lambda^{-1} m$. However, their argument is really independent of such assumptions, as we shall show below. We work from first principles to clearly display all assumptions made.

We consider the motion of a charged point particle in an electromagnetic field. The whole system, i.e. particle plus electromagnetic field, is placed into a cosmological FRW-spacetime with flat ($k = 0$) spatial geometry. The spacetime metric reads

$$g = c^2 dt^2 - a^2(t)(dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)). \quad (37)$$

We introduce conformal time, t_c , via

$$t_c = f(t) := \int_k^t \frac{dt'}{a(t')}, \quad (38)$$

by means of which we can write (37) in a conformally flat form, where η denotes the flat Minkowski metric:

$$g = a_c^2(t_c) \underbrace{\{c^2 dt_c^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2)\}}_{\eta}. \quad (39)$$

Here we wrote a_c to indicate that we now expressed the expansion parameter a as function of t_c rather than t , i.e.

$$a_c := a \circ f^{-1}. \quad (40)$$

For example, if $a(t) = \sigma t^n$ ($0 < n < 1$), then we can choose $k = 0$ in (38) and have

$$t_c = f(t) = \int_0^t \frac{dt'}{\sigma t'^n} = \frac{t^{1-n}}{\sigma(1-n)}, \quad (41)$$

so that

$$t = f^{-1}(t_c) = [(1-n)\sigma t_c]^{1/(1-n)}, \quad (42)$$

and therefore

$$a_c(t_c) = \alpha t_c^{n/(1-n)}, \quad \text{where } \alpha := [(1-n)^n \sigma]^{1/(1-n)}. \quad (43)$$

The electromagnetic field is characterized by the tensor $F_{\mu\nu}$, comprising electric and magnetic fields:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_n/c \\ -E_m/c & -\varepsilon_{mnj} B_j \end{pmatrix}. \quad (44)$$

In terms of the electromagnetic four-vector potential, $A_\mu = (\varphi/c, -\vec{A})$, one has

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \quad (45)$$

so that, as usual, $\vec{E} = -\vec{\nabla}\phi - \dot{\vec{A}}$. The expression for the four-vector of the Lorentz-force of a particle of charge e moving in the field $F_{\mu\nu}$ is $e F^\mu{}_\alpha u^\alpha$, where u is the particle's four velocity.

The equations of motion for the system Particle + EM-Field follow from an action which is the sum of the action of the particle, the action for its interaction with the electromagnetic field, and the action for the free field, all placed in the background (37). Hence we write:

$$S = S_P + S_I + S_F, \quad (46)$$

where

$$S_P = -mc^2 \int_z d\tau = -mc \int \sqrt{g(z', z')} d\lambda, \quad (47a)$$

$$\begin{aligned} S_I &= -e \int_z A_\mu dx^\mu = -e \int A_\mu(z(\lambda)) z'^\mu d\lambda \\ &= - \int d^4x A_\mu(x) \int d\lambda e \delta^{(4)}(x - z(\lambda)) z'^\mu, \end{aligned} \quad (47b)$$

$$S_F = \frac{-1}{4} \int d^4x \sqrt{-\det g} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} = \frac{-1}{4} \int d^4x \eta^{\mu\alpha} \eta^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}. \quad (47c)$$

Here λ is an arbitrary parameter along the worldline $z : \lambda \mapsto z(\lambda)$ of the particle, and z' the derivative $dz/d\lambda$. The differential of the eigentime along this worldline is

$$d\tau = \sqrt{g(z', z')} d\lambda = \sqrt{g_{\mu\nu}(z(\lambda)) \frac{dz^\mu}{d\lambda} \frac{dz^\nu}{d\lambda}} d\lambda. \quad (48)$$

It is now important to note that 1) the background metric g does not enter (47b) and that (47c) is conformally invariant (in 4 spacetime dimensions only!). Hence the expansion factor, $a(t_c)$, does not enter these two expressions. For this reason we could write (47c) in terms of the flat Minkowski metric, though it should be kept in mind that the time coordinate is now given by conformal time t_c . This is *not* the time read by standard clocks that move with the cosmological observers, which rather show the cosmological time t (which is the proper time along the geodesic flow of the observer field $X = \partial/\partial t$).

The situation is rather different for the action (47a) of the particle. Its variational derivative with respect to $z(\lambda)$ is

$$\frac{\delta S_p}{\delta z^\mu(\lambda)} = -mc \left\{ \frac{\frac{1}{2}g_{\alpha\beta,\mu} z'^\alpha z'^\beta}{\sqrt{g(z', z')}} - \frac{d}{d\lambda} \left[\frac{g_{\mu\alpha} z'^\alpha}{\sqrt{g(z', z')}} \right] \right\}. \quad (49)$$

We now introduce the *conformal proper time*, τ_c , via

$$d\tau_c = (1/c) \sqrt{\eta(z', z')} d\lambda = (1/ca) \sqrt{g(z', z')} d\lambda. \quad (50)$$

We denote differentiation with respect to τ_c by an overdot, so that e.g. $z'/\sqrt{g(z', z')} = \dot{z}/ca$. Using this to replace z' by $\dot{z}\sqrt{g(z', z')}/ca$ and also g by $a^2\eta$ in (49) gives

$$\frac{\delta S_p}{\delta z^\mu(\lambda)} = \frac{\sqrt{g(z', z')}}{ac} ma \{ \eta_{\mu\alpha} \ddot{z}^\alpha + P_\mu^\alpha \phi_{,\alpha} \} \quad (51)$$

where we set

$$a =: \exp(\phi/c^2) \quad \text{and} \quad P_\mu^\alpha := -\delta_\mu^\alpha + \frac{\dot{z}^\alpha \dot{z}^\nu}{c^2} \eta_{\nu\mu}. \quad (52)$$

Recalling that $\delta S_p = \int \frac{\delta S_p}{\delta z^\mu(\lambda)} \delta z^\mu d\lambda = \int \frac{\delta S_p}{\delta z^\mu(\tau_c)} \delta z^\mu d\tau_c$ and using (50), (51) is equivalent to

$$\frac{\delta S_p}{\delta z^\mu(\tau_c)} = ma(\ddot{z}^\alpha + P_\mu^\alpha \phi_{,\alpha}), \quad (53)$$

where from now on we agree to raise and lower indices using the Minkowski metric, i.e. $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ in Minkowski inertial coordinates.

Writing (47b) in terms of the conformal proper time and taking the variational derivative with respect to $z(\tau_c)$ leads to $\delta S_I/\delta z^\mu(\tau_c) = -eF_{\mu\alpha}\dot{z}^\alpha$, so that

$$\frac{\delta S}{\delta z^\mu(\tau_c)} = ma(\ddot{z}_\mu + P_\mu^\alpha \phi_{,\alpha}) - eF_{\mu\alpha}\dot{z}^\alpha. \quad (54)$$

The variational derivative of the action with respect to the vector potential A is

$$\frac{\delta S}{\delta A_\mu(x)} = \partial_\alpha F^{\mu\alpha}(x) - e \int d\tau_c \delta(x - z(\tau_c)) \dot{z}^\mu(\tau_c). \quad (55)$$

Equations (54) and (55) show that the fully dynamical problem can be treated as if it were situated in static flat space. The field equations that follow from (55) are just the same as in Minkowski space. Hence we can calculate the Coulomb field as usual. On

the other hand, the equations of motion receive two changes from the cosmological expansion term: the first is that the mass m is now multiplied with the (time-dependent!) scale factor a , the second is an additional scalar force induced by a . Note that all spacetime dependent functions on the right hand side are to be evaluated at the particle's location $z(\tau_c)$, whose fourth component corresponds to ct_c . Hence, writing out all arguments and taking into account that the time coordinate is t_c , we have for the equation of motion

$$\ddot{z}^\mu = \frac{e}{ma_c(z^0/c)} F^\mu{}_\alpha(z) \dot{z}^\alpha - (-c^2 \eta^{\mu\alpha} + \dot{z}^\mu \dot{z}^\alpha) \partial_\alpha \ln a_c(z^0/c) \quad (56a)$$

$$= \frac{e}{ma_c(z^0/c)} F^\mu{}_\alpha(z) \dot{z}^\alpha - (-c\eta^{\mu 0} + \dot{z}^\mu \dot{z}^0/c) a'_c(z^0/c)/a_c(z^0/c), \quad (56b)$$

where a'_c is the derivative of a_c .

So far no approximations were made. Now we write $\dot{z}^\mu = \gamma(c, \vec{v})$, where \vec{v} is the derivative of \vec{z} with respect to the conformal time t_c , henceforth denoted by a prime, and $\gamma = 1/\sqrt{1 - v^2/c^2}$. Then we specialize to slow motions, i.e. neglect effects of quadratic or higher powers in v/c (special relativistic effects). For the spatial part of (56b) we get

$$\ddot{\vec{z}} + \vec{z}' (a'_c/a_c) = \frac{e}{ma_c} (\vec{E} + \vec{z}' \times \vec{B}), \quad (57)$$

where we once more recall that the spatial coordinates used here are the comoving (i.e. conformal) ones and the electric and magnetic fields are evaluated at the particle's position $\vec{z}(t_c)$.

From the above equation we see that the effect of cosmological expansion in the conformal coordinates shows up in two ways: first in a time dependence of the mass which scales with a_c , and, second, in the presence a friction term. Let us, for the moment, neglect the friction term. In the adiabatic approximation, which is justified if typical time scales of the problem at hand are short compared to the world-age (corresponding to small ε_2 in (36b)), the time-dependent mass term leads to a time varying radius in comoving (or conformal) coordinates of $r(t_c) \propto 1/a_c(t_c)$. Hence the physical radius (given by the cosmological geodesically spatial distance), $r_* = a_c r$, stays constant in this approximation. In this way Dicke & Peebles concluded in [48] that electromagnetically bound systems do not feel *any* effect of cosmological expansion.

Let us now look at the effect of the friction term which the analysis of Dicke & Peebles neglects. It corresponds to the decelerating force $-\vec{v} a'_c/a_c$ which e.g. for the simple power-law expansion (43) becomes

$$-\vec{v} \frac{n}{n-1} t_c^{-1} = -\vec{v} n \sigma t^{n-1}. \quad (58)$$

Clearly it must cause any stationary orbit to decay. For example, as a standard first-order perturbation calculation shows, a circular orbit of radius r and angular frequency (with respect to conformal time t_c) ω_c will suffer a relative decay per revolution of

$$\left. \frac{\Delta r}{r} \right|_{\text{revol.}} = -\frac{a'_c/a_c}{3\omega_c}. \quad (59)$$

Recall that this is an equation in the (fictitious) Minkowski space obtained after rescaling the physical metric. However, it equates two scale invariant quantities. Indeed, the

relative length change $\Delta r/r$ is certainly scale invariant and so is the relative length change per revolution. On the right hand side we take the quotient of two quantities which scale like an inverse time. In physical spacetime, coordinatized by cosmological time t , the right hand side becomes $-H/3\omega$, where as usual $H = \dot{a}/a$ and ω is the angular frequency with respect to t . Hence the relative radial decay in physical space is of the order of the ratio between the orbital period and the inverse Hubble constant ('world-age'), i.e. of order ε_2 (cf.(36b)).

Since the friction term contributes to the leading-order effect of cosmological expansion, we conclude that the argument of Dicke & Peebles, which neglects this term, is not sufficient to estimate such effects.

4.2 Equations of motion in the physical coordinates

We now show that (5) is indeed arrived at if the friction term is consistently taken into account. To see this we merely need to rewrite equation (57) in terms of the physical coordinates given by the cosmological time t and the cosmological geodesic spatial distance $r_* := a(t)r$. We have $dt_c/dt = 1/a$ and the spatial geodesic coordinates are $\vec{y} := a(t)\vec{z}$. Denoting by an overdot the time derivative with respect to t , the left hand side of (57) becomes

$$\vec{z}'' + \vec{z}' (a'_c/a_c) = a \ddot{\vec{y}} - \ddot{a} \vec{y}. \quad (60)$$

This shows that the friction term in the unphysical coordinates becomes, in the physical coordinates, the familiar acceleration term (2) due to the Hubble-law. Dividing by a equation (57) and inserting $\vec{E}(\vec{z}) = Q\vec{z}/|\vec{z}|^3$ and $\vec{B}(\vec{z}) = 0$, we get

$$\ddot{\vec{y}} - \vec{y} (\ddot{a}/a) = \frac{eQ}{m|\vec{y}|^3} \vec{y}. \quad (61)$$

Finally, introducing polar coordinates in the orbital plane we exactly get (5).

5 Kinematical effects

It has been suggested in [71] and again in [72] that there may be significant *kinematical* effects that may cause apparent anomalous acceleration of spacecraft orbits in an expanding cosmological environment. More precisely it was stated that there is an additional acceleration of magnitude $Hc \approx 0.7 \cdot 10^{-9} \text{m/s}^2$, which is comparable to the measured anomalous acceleration of the Pioneer spacecrafts. The cause of such an effect lies in the way one actually measures spatial distances and determines the clock readings they are functions of (a trajectory is a 'distance' for each given 'time'). The point is this: equations of motions give us, for example, simultaneous (with respect to cosmological time) spatial geodesic distances as functions of cosmological time. This is what we implicitly did in the Newtonian analysis. But, in fact, spacecraft ranging is done by exchanging electromagnetic signals. The notion of spatial distance as well as the notion of simultaneity introduced thereby is *not* the same. Hence the analytical expression of the 'trajectory' so measured will be different.

To us this seems an important point and the authors of [71] and [72] were well justified to draw proper attention to it. However, we will now explain why we do not arrive at their conclusion. Again we take care to state all assumptions made.

5.1 Local Einstein-simultaneity in general spacetimes

We consider a general Lorentzian manifold (\mathcal{M}, g) as spacetime. Our signature convention is $(+, -, -, -)$ and ds is taken to have the unit of length. The differential of eigentime is $d\tau = ds/c$. In general coordinates $\{x^\mu\}$, the metric reads

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{tt} dt^2 + 2g_{ta} dt dx^a + g_{ab} dx^a dx^b. \quad (62)$$

The observer at fixed spatial coordinates is given by the vector field (normalized to $g(X, X) = c^2$)

$$X = \frac{c}{\sqrt{g_{tt}}} \frac{\partial}{\partial t}. \quad (63)$$

Consider the light cone with vertex $p \in \mathcal{M}$; one has $ds^2 = 0$, which allows to solve for dt in terms of the dx^a (all functions g_{ab} are evaluated at p , unless noted otherwise):

$$dt_{1,2} = -\frac{g_{ta}}{g_{tt}} dx^a \pm \sqrt{\left(\frac{g_{ta}g_{tb}}{g_{tt}^2} - \frac{g_{ab}}{g_{tt}}\right) dx^a dx^b}. \quad (64)$$

The plus sign corresponds to the future light-cone at p , the negative sign to the past light cone. An integral line of X in a neighbourhood of p cuts the light cone in two points, q_+ and q_- . If t_p is the time assigned to p , then $t_{q_+} = t_p + dt_1$ and $t_{q_-} = t_p + dt_2$. The coordinate-time separation between these two cuts is $t_{q_+} - t_{q_-} = dt_1 - dt_2$, corresponding to a proper time $\sqrt{g_{tt}}(dt_1 - dt_2)/c$ for the observer X . This observer will associate a *radar-distance* dl_* to the event p of $c/2$ times that proper time interval, that is:

$$dl_*^2 = h = \left(\frac{g_{ta}g_{tb}}{g_{tt}} - g_{ab}\right) dx^a dx^b. \quad (65)$$

The event on the integral line of X that the observer will call Einstein-synchronous with p lies in the middle between q_+ and q_- . Its time coordinate is in first-order approximation given by $\frac{1}{2}(t_{q_+} + t_{q_-}) = t_p + \frac{1}{2}(dt_1 + dt_2) = t_p + dt$, where

$$dt := \frac{1}{2}(dt_1 + dt_2) = -\frac{g_{ta}}{g_{tt}} dx^a. \quad (66)$$

This means the following: the Integral lines of X are parameterized by the spatial coordinates $\{x^a\}_{a=1,2,3}$. Given a point p , specified by the orbit-coordinates x_p^a and the time-coordinate t_p , we consider a neighbouring orbit of X with orbit-coordinates $x_p^a + dx^a$. The event on the latter which is Einstein synchronous with p has a time coordinate $t_p + dt$, where dt is given by (66), or equivalently

$$\theta := dt + \frac{g_{ta}}{g_{tt}} dx^a = 0. \quad (67)$$

Using a differential geometric language we may say that Einstein simultaneity defines a *distribution* $\theta = 0$.

The metric (62) can be written in terms of the radar-distance metric h (65) and the simultaneity 1-form θ as follows:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{tt} \theta^2 - h, \quad (68)$$

showing that the radar-distance is just the same as the Einstein-simultaneous distance. A curve γ in \mathcal{M} intersects the flow lines of X perpendicularly iff $\theta(\dot{\gamma}) = 0$, which is just the condition that neighbouring clocks along γ are Einstein synchronized.

5.2 Application to isotropic cosmological metrics

We consider isotropic cosmological metrics. In what follows we drop for simplicity the angular dimensions. Hence we consider metrics of the form

$$ds^2 = c^2 dt^2 - a(t)^2 dr^2. \quad (69)$$

The expanding observer field is

$$X = \frac{\partial}{\partial t}. \quad (70)$$

The Lagrangian for radial geodesic motion is $L = \frac{1}{2}(c^2 \dot{t}^2 - a^2 \dot{r}^2)$, leading to the only non-vanishing Christoffel symbols

$$\Gamma_{rr}^t = \frac{a\dot{a}}{c^2}, \quad \Gamma_{tr}^r = \frac{\dot{a}}{a} =: H. \quad (71)$$

Hence X is geodesic, since

$$\nabla_X X = \Gamma_{tt}^\mu \partial_\mu = 0. \quad (72)$$

On a hypersurface of constant t the radial geodesic distance is given by $ra(t)$. Making this distance into a spatial coordinate, r_* , we consider the coordinate transformation

$$t \mapsto t_* := t, \quad r \mapsto r_* := a(t)r. \quad (73)$$

The field $\partial/\partial t_*$ is given by

$$\frac{\partial}{\partial t_*} = \frac{\partial}{\partial t} - Hr \frac{\partial}{\partial r}. \quad (74)$$

In contrast to (70), whose flow connects co-moving points of constant coordinate r , the flow of (74) connects points of constant geodesic distances, as measured in the surfaces of constant cosmological time. This could be called *cosmologically instantaneous geodesic distance*. It is now very important to realize that this notion of distance is not the same as the radar distance that one determines by exchanging light signals in the usual (Einsteinian) way. Let us explain this in detail:

From (73) we have $adr = dr_* - r_* H dt$, where $H := \dot{a}/a$ (Hubble parameter). Rewriting the metric (69) in terms of t_* and r_* yields

$$\begin{aligned} ds^2 &= c^2(1 - (Hr_*/c)^2) dt_*^2 - dr_*^2 + 2Hr_* dt_* dr_* \\ &= \underbrace{c^2 \left\{ 1 - (Hr_*/c)^2 \right\}}_{g_{t_*t_*}} \underbrace{\left\{ dt_* + \frac{Hr_*/c^2}{1 - (Hr_*/c)^2} dr_* \right\}^2}_{\theta} - \underbrace{\frac{dr_*^2}{1 - (Hr_*/c)^2}}_h, \quad (75) \end{aligned}$$

Hence the differentials of radar-distance and time-lapse for Einstein-simultaneity are given by

$$dl_* = \frac{dr_*}{\sqrt{1 - (Hr_*/c)^2}}, \quad (76a)$$

$$dt_* = -\frac{Hr_*/c^2}{1 - (Hr_*/c)^2} dr_*. \quad (76b)$$

Let the distinguished observer (us on earth) now move along the geodesic $r_* = 0$. Integration of (76) from $r_* = 0$ to some value r_* then gives the radar distance l_* as well as the time lapse Δt_* as functions of the cosmologically simultaneous geodesic distance r_* :

$$l_* = (c/H) \sin^{-1}(Hr_*/c) \approx r_* \left\{ 1 + \frac{1}{6}(Hr_*/c)^2 + O(3) \right\} \quad (77a)$$

$$\Delta t_* = (1/2H) \ln(1 - (Hr_*/c)^2) \approx (r_*/c) \left\{ -\frac{1}{2}(Hr_*/c) + O(2) \right\} \quad (77b)$$

Combining both equations in (77) allows to express the time-lapse in terms of the radar-distance:

$$\Delta t_* = H^{-1} \ln(\cos(Hl_*/c)) \approx (l_*/c) \left\{ -\frac{1}{2}(Hl_*/c) + O(2) \right\}. \quad (78)$$

Now, suppose a satellite S moves on a worldline $r_*(t_*)$ in the neighbourhood of our worldline $r_* = 0$. Assume that we measure the distance to the satellite by radar coordinates. Then instead of the value r_* we would use l_* and instead of the argument t_* we would assign the time $t_* - \Delta t_*$ which corresponds to the value of cosmological time at that event on our worldline that is Einstein synchronous to the event (t_*, r_*) ; see Figure 2.

Hence we have

$$l_*(t_*) = (c/H) \sin^{-1}\{r_*(t_* + \Delta t_*)H/c\} \quad (79a)$$

$$\approx r_* - \frac{1}{2}(v/c)(Hc)(r_*/c)^2, \quad (79b)$$

where (79b) is (79a) to leading order and all quantities are evaluated at t_* . We set $v = \dot{r}_*$.

To see what this entails we Taylor expand in t_* around $t_* = 0$ (just a convenient choice):

$$r_*(t_*) = r_0 + v_0 t_* + \frac{1}{2} a_0 t_*^2 + \dots \quad (80)$$

and insert in (79b). This leads to

$$l_*(t_*) = \tilde{r}_0 + \tilde{v}_0 t_* + \frac{1}{2} \tilde{a}_0 t_*^2 + \dots, \quad (81)$$

where,

$$\tilde{r}_0 = r_0 - (Hc) \frac{1}{2} (v_0/c) (r_0/c)^2 \quad (82)$$

$$\tilde{v}_0 = v_0 - (Hc) (v_0/c)^2 (r_0/c) \quad (83)$$

$$\tilde{a}_0 = a_0 - (Hc) \left\{ (v_0/c)^3 + (r_0/c)(v_0/c)(a_0/c) \right\} \quad (84)$$

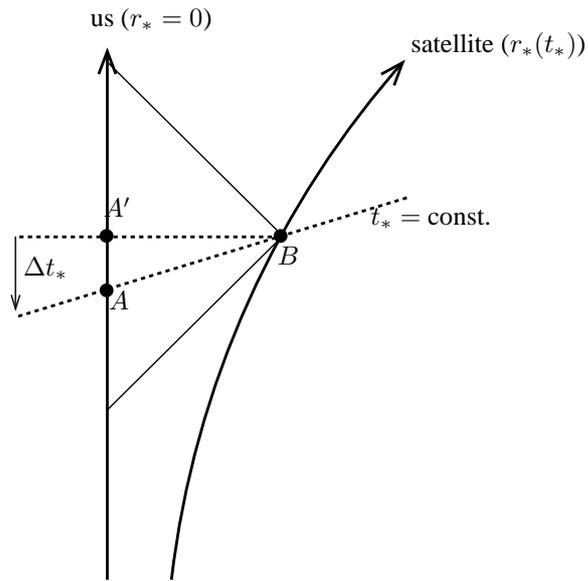


Figure 2: The observer ('us') moves on the geodesic worldline $r_* = 0$ and measures the spatial distance to a satellite by exchanging electromagnetic signals (radar coordinates). The satellite's worldline is analytically described by a function $r_*(t_*)$. The surface of constant cosmological time t_* (tilted dashed line) intersects our worldline at A and the satellite's worldline at B . The coordinate r_* corresponds to the geodesic distance in that hypersurface of constant cosmological time, i.e. to the length of \overline{AB} as measured in the spacetime geometry. However, using radar coordinates, the observer defines B to be simultaneous to his event A' and attributes to it the distance l_* . Hence instead of $r_*(t_*)$ he uses $l_*(r_*(t_* - \Delta t_*))$, where $l_*(r_*)$ is given by (77a). This leads to (79).

These are, in quadratic approximation, the sought-after relations between the quantities measured via radar tracking (tilded) and the quantities which arise in the (improved) Newtonian equations of motion (not tilded).

The last equation (84) shows that there is an apparent inward pointing acceleration, given by Hc times the $(v/c)^3 + \dots$ term in curly brackets. The value of Hc is indeed of the same order of magnitude as the anomalous Pioneer acceleration, as emphasized in [71, 72]. However, in contrast to these authors, we do get the additional term in curly brackets, which in case of the Pioneer spacecraft suppresses the Hc term by 13 orders of magnitude! Hence, according to our analysis, and in contrast to what is stated in [71, 72], there is no significant kinematical effect resulting from the distinct simultaneity structures inherent in radar and cosmological coordinates. We should stress, however, that this verdict is strictly limited to our interpretation of what the kinematical effect actually consists in, which is most concisely expressed in (79).⁷

6 Summary and outlook

We think it is fair to say that there are no theoretical hints that point towards a *dynamical* influence of cosmological expansion comparable in size to that of the anomalous acceleration of the Pioneer spacecrafts. There seems to be no controversy over this point, though for completeness it should be mentioned that according to a recent suggestion [69] it might become relevant for future missions like LATOR. This suggestion is based on the model of Gautreau [14] which, as already mentioned in Section 3, we find hard to relate to the problem discussed here. Rather, as the (\ddot{a}/a) -improved Newtonian analysis in Section 2 strongly suggests, there is no genuine relativistic effect coming from cosmological expansion at the level of precision reached e.g. by the LATOR mission.

On the other hand, as regards *kinematical* effects, the situation is less unanimous. It is very important to unambiguously understand what is meant by ‘mapping out a trajectory’, i.e. how to assign ‘times’ and ‘distances’. Eventually we compare a functional relation between ‘distance’ and ‘time’ with observed data. That relation is obtained by solving some equations of motion and it has to be carefully checked whether the methods by which the tracking data are obtained match the interpretation of the coordinates in which the analytical problem is solved. In our way of speaking dynamical effects really influence the worldline of the object in question, whereas kinematical effects change the way in which one and the same worldline is mapped out from another worldline representing the observer.

The latter problem especially presents itself in a time dependent geometry of space-time. Mapping out a trajectory then becomes dependent on ones definition of ‘simultaneity’ and ‘simultaneous spatial distance’ which cease to be unique. An intriguing suggestion has been made [71, 72] that the PA is merely a result of such an ambiguity.

⁷ Our equation (78) corresponds to equation (10) of [71]. From it the authors of [71] and [72] immediately jump to the conclusion that there is ‘an effective residual acceleration directed toward the centre of coordinates; its constant value is Hc ’. We were unable to understand how this conclusion is reached. Our interpretation of the meaning of (79) does not support this conclusion.

However, our analysis suggests that no significant relativistic effects result within the Solar System, over and above those already taken into account, as e.g. the Shapiro time delay.

What has been said so far supports the view that there is no interesting impact of cosmological expansion on the specific problem of *satellite navigation* in the Solar System. However, turning now to a more general perspective, the problem of how local inhomogeneities on a larger scale affect, and are affected by, cosmological expansion is of utmost importance. Many scientific predictions concerning cosmological data rely on computations within the framework of the standard homogeneous and isotropic models, without properly estimating the possible effects of local inhomogeneities. Such an estimation would ideally be based on an exact inhomogeneous solution to Einstein's equations, or at least a fully controlled approximation to such a solution. The dynamical and kinematical impact of local inhomogeneities might essentially influence our interpretation of cosmological observations. As an example we mention recent serious efforts to interpret the same data that are usually taken to prove the existence of a positive cosmological constant Λ in a context with realistic inhomogeneities where $\Lambda = 0$; see [46] and [57].

To indicate possible directions of research, we stress again that there are several approaches to the problem of how to rigorously combine an idealized local inhomogeneity—a single star in the most simple case—with an homogeneous and isotropic cosmological background. We mentioned that of Einstein & Straus [7] and its refinement by Schücking [33], that of Gautreau [14], that by Bonnor [43, 44, 45, 3], and especially the classic work by McVittie [58] that was later elaborated on by Hogan [19] and properly interpreted by the penetrating analysis of Nolan [29, 30, 31]. Nolan also showed that only in the case $k = 0$ does McVittie's solution represent a central mass embedded into a cosmological background. The problem is that, due to the non-linearity of Einstein's equations, the spherical inhomogeneity does not show just as an addition to the background. Hence a notion of quasi-local mass has to be employed in order to theoretically detect local mass abundance. However, as is well known, it is a notoriously difficult problem in General Relativity to define a physically appropriate notion of quasi-local mass. Workable definitions only exist in special circumstances, as for example in case of spherical symmetry, where the concept of Misner-Sharp mass can be employed, as explained in Section 3.1.

As a project for future research we therefore suggest to further probe and develop applications of the spatially flat ($k = 0$) McVittie solution, taking due account of recent progress in our theoretical understanding of it. As a parallel development, the implications of Gautreau's model should be developed to an extent that allows its comparison with those of McVittie.

A Additional material

In this section we collect some background information which was implicitly used throughout the text.

A.1 A simple estimate of the dynamical effect of cosmological expansion

The radial acceleration due to the Newtonian Sun attraction is given by:

$$\ddot{r}|_{\text{Sun}} = -\frac{GM_{\odot}}{r^2} \approx -\frac{60}{(r/10 \text{ AU})^2} 10^{-6} \text{ m/s}^2. \quad (85)$$

The geodesic distance, r_* , between two freely falling bodies in an expanding FRW universe (37), measured on a hypersurface of constant cosmological time, varies in time according to the Hubble-law $\dot{r}_* = Hr_*$. The related ‘acceleration’ is then $\ddot{r}_* = \dot{r}_*H + r_*\dot{H} = r_*(H^2 + \dot{H}) = r_*\ddot{a}/a = -qH^2r_*$. At the present time (see Section A.3) we have (suppressing the asterisk):

$$\ddot{r}|_{\text{cosm. acc.}} = \frac{\ddot{a}}{a}r \approx 4(r/10 \text{ AU}) 10^{-24} \text{ m/s}^2. \quad (86)$$

This naive derivation of the *dynamical* effect of the cosmological expansion may be confirmed by the fully general relativistic treatment, as showed in Section 3. Notice that, according to our measurements, the universe is presently in a phase of accelerated expansion, hence (86) results in an acceleration pointing away from the Sun. This is in the *opposite* direction of the Pioneer-effect (92) and also smaller by 14 orders of magnitude.

A.2 The Schücking radius for different astronomical scales

In the following we evaluate the radius of the Schücking vacuole (10) for various characteristic central masses. Using as cosmological matter density $\rho_m = \Omega_m\rho_c$ (see Appendix A.3), we get:

- **Solar system scale:** $r_S(M_{\odot}) = 570 \text{ ly}$, which is much larger than the average distance between stars in the Milky Way, being about 10 ly.
- **Galaxy scale:** $r_S(M_{\text{MW}}) = 3 - 4 \text{ Mly}$, which is again too big since this would also include other galaxies such as the Large and the Small Magellanic Cloud, as well as several dwarf galaxies.
- **Cluster scale:** $r_S(M_{\text{LG}}) = 5 - 7 \text{ Mly}$, which is just about the threshold since the nearest object not belonging to this cluster is NGC 55 (a galaxy belonging to the Sculptor Group) about 5 Mly away.
- **Supercluster scale:** $r_S(M_{\text{VSC}}) = 40 \text{ Mly}$, which is inside the Virgo Supercluster whose radius is about 100 Mly.

This shows that the Einstein-Straus matching works at best from and above cluster scale.

A.3 Some astronomical and cosmological data

For the convenience of the readers, we collect some relevant numerical information.

Length units

$$1 \text{ AU} = 149.6 \cdot 10^6 \text{ km} = 1.5 \cdot 10^{11} \text{ m} = 492 \text{ ls} = 8.2 \text{ lmin} = 1.58 \cdot 10^{-5} \text{ ly}$$

$$1 \text{ pc} = 3.26 \text{ ly}$$

Time units

$$1 \text{ yr} = 3.1 \cdot 10^7 \text{ s}$$

$$\text{Age of the Universe} \approx 13.7 \cdot 10^9 \text{ yr} = 4.32 \cdot 10^{17} \text{ s}$$

Velocity units

$$1 \text{ AU/yr} = 4.74 \text{ km/s}$$

Mass units

$1 M_{\odot} = 2 \cdot 10^{30} \text{ kg}$ (= sometimes referred to as twice the mass of the average star in the Milky Way)

The Universe

Our galaxy: Milky Way (MW)

Number of stars in the MW: $2 \dots 4 \cdot 10^{11}$

$$M_{\text{MW}} = 2 \dots 4 \cdot 10^{11} M_{\odot}$$

Diameter: 100 kly

Average distance between stars in the MW: 10 ly

Nearest galaxy: Large Magellanic Cloud. Mass: $10^{10} M_{\odot}$, distance: 170 kly, diameter: 30 kly

Our cluster: Local Group (LG)

Number of stars in the LG: $7 \cdot 10^{11}$

$$M_{\text{LG}} = 7 \dots 20 \cdot 10^{11} M_{\odot}$$

Diameter: 10 Mly

Nearest clusters: Sculptor Group (distance: 10 Mly) and Maffei 1 Groups (distance: 10 Mly).

Nearest galaxy: NGC55 (Sculptor Group), distance: 5 Mly.

Our supercluster: Virgo Supercluster (VSC)

Number of stars in the VSC: $2 \cdot 10^{14}$

$$M_{\text{VSC}} = 1 \cdot 10^{15} M_{\odot}$$

Diameter: 200 Mly

Nearest supercluster: Centaurus (distance: 200 Mly)

Nearest cluster: A3526 (Centaurus Supercluster), distance: 142 Mly

Cosmology data

Hubble parameter (today): $H_0 = h_0 \cdot 100 \text{ km s}^{-1} \text{ Mpc}^{-1} = h_0 / (3.08 \cdot 10^{17} \text{ s})$,
where $h_0 = 0.7$

Critical density (today): $\rho_c = 3H_0^2 / 8\pi G = h_0^2 \cdot 1.89 \cdot 10^{-29} \text{ g/cm}^3$

Definition of the cosmological parameters:

$$\Omega_m := \rho_m / \rho_c, \quad (87)$$

$$\Omega_\Lambda := \rho_\Lambda / \rho_c = \Lambda / 3H_0^2, \quad (88)$$

$$\Omega_k := -kc^2 / (H_0 a_0)^2. \quad (89)$$

One has $\Omega_m + \Omega_\Lambda + \Omega_k = 1$ (cosmological triangle), where todays values are given by:

$$(\Omega_m, \Omega_\Lambda, \Omega_k) \approx (1/3, 2/3, 0). \quad (90)$$

Thus from $q_0 = (1/2)\Omega_m - \Omega_\Lambda$ it follows that $q_0 \approx -1/2$.

Pioneer 10 and 11 data

The following data are taken from [74].

	P10	P11
Launch	2 Mar 1972	5 Apr 1973
Planetary encounters	Jupiter: 4 Dec 1973	Jupiter: 2 Dec 1974 Saturn: 1 Sep 1979
Tracking data	3 Jan 1987 – 22 Jul 1998	5 Jan 1987 – 1 Oct 1990
Distance from the Sun	40 AU – 70 AU	22.4 AU – 31.7 AU
Light round-trip time	11 h – 19 h	6 h – 9 h
Radial velocity	13.1 Km/s – 12.6 Km/s	N.A.

Table 1: Some orbital data of the Pioneer 10 and 11 spacecrafts.

Tracking system

Uplink frequency as received from Pioneer (approx.): $\nu_{u,2} = 2.11 \text{ GHz}$.

Downlink frequency emitted from Pioneer: $\nu_{d,2} = T\nu_{u,2} = 2.292 \text{ GHz}$.

Spacecraft transponder turnaround ratio: $T = 240/221$.

Measured effect

Measured is an almost constant residual (meaning after subtraction of all the known effects) frequency drift of the received tracking signal. The drift is a blue-shift at the constant rate

$$\dot{\nu} = (5.99 \pm 0.01) 10^{-9} \text{ Hz/s} \quad (91)$$

which, if interpreted as a special-relativistic Doppler shift, can be rewritten as an acceleration pointing towards the Earth (or Sun) of modulus

$$a_P = (8.74 \pm 1.33) 10^{-10} \text{ m/s}^2. \quad (92)$$

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