Analogue Transformation Acoustics

Covariant Methodology in Transformational Acoustics

Final Report

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Analogue Transformational Acoustics: An Alternative Theoretical Framework for Acoustic Metamaterials
or
Covariant Methodology in Transformational Acoustics

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Abstract

The work of Ariadne study reference 11-1301 is described. Focus was given to developing a formal theory of transformation acoustics that analogued the exact theory of transformation electromagnetics. This was developed both for simple isotropic media and for so-called pentamode acoustic media. In addition a generalised ray-based transformation theory that provided a simple, but highly flexible, recipe for making transformation-based devices, such as an acoustic cloak. Finally, an analogue acoustic theory that mimics the structure of covariant electromagnetism is discussed.

1 Introduction

The programme of actualizing point deformations in media [1],[2] has opened up many avenues of research, particularly in electromagnetics. Conceptually, the idea is beautiful: think of a deformation that maps points in space to new points in space[1]. For a given domain of physics (electromagnetism, acoustics, etc.), enquire as to how the relevant medium properties should be modified (anisotropy and inhomogeneity) so as to reposition the field quantities in question (e.g. electromagnetic fields) to ones located at the new points. If this question is answered, then we have a recipe for 'designer' deformation, the most celebrated example of which is the electromagnetic cloak [3].

For electromagnetism, the prescription is known to be exact, i.e. *all* properties of Maxwell's equations, including both near- and far-field attributes, are inherited by the transformed fields, irrespective of wavelength, obstacle sizes and shapes. Whatever the technological challenges in producing the required electromagnetic medium, if it can be made, then Maxwell's equations are mapped *exactly* onto the new domain. The linear rays of geometrical optics in a homogeneous medium are then mapped precisely to the curved paths defined by the mapping.

Why the programme is so outstandingly successful in electromagnetism, and not in other areas of physics, is still a matter of debate and ongoing research. The fact that Maxwell's equations admit a covariant, or co-ordinate independent formulation is undoubtedly significant. Notably, the most concerted mathematical studies of transformation electromagnetics has been carried out by Thompson et al [4, 5]. Acoustics, the focus of this current report, has been the most explored area for transformation theory besides electromagnetism. Milton *et al* [6] have shown that in general an anisotropic mass density is required for elastodynamic equations that are the natural counterparts of Maxwell's equations to remain invariant under spatial transformation. However, the equations Milton uses as acoustic counterparts of Maxwell are posed in terms of ordinary, rather than covariant, derivatives. As a consequence, Milton's equations of motion are not inherently form invariant and thus do not admit the full range of transformation-type solutions that are possible in electromagnetism. One of our key objectives in this report is to provide a properly covariant framework for acoustics. In two-dimensions, the acoustic equations in a fluid are isomorphic to the

[1]We draw a distinction between *deformations* that reposition points in a space, to *coordinate transformations* that merely relabel points in the space. Although deformations can be related to a coordinate relabelling, the converse is not true in general.

single polarization Maxwell equations in a way that also preserves boundary conditions as was shown by Cummer *et al* [7]. The form invariance of Helmholz equations under general curvilinear transformations was studied by Greenleaf *et al* [8] who showed in particular that the scalar wave equation for the pressure in an inviscid (i.e. non-viscous) medium can be cast as

$$\frac{1}{\sqrt{|\bar{g}|}}\frac{\partial}{\partial x_i}\left(\sqrt{|\bar{g}|}\bar{g}^{ij}\frac{\partial p}{\partial x_j}\right) = \frac{\partial^2 p}{\partial t^2}\ , \tag{1}$$

where $\bar{g}_{ij}$ is a metric induced by the proposed transformation. Acoustic cloaking in inviscid media for which the mass density is anisotropic and the bulk modulus varies, have become known as *Inertial Cloaks* [9]. Norris showed that inertial cloaking requires the cloaking medium to have a locally infinite mass density. If instead more general anisotropic media are considered in which the bulk modulus is also anisotropic, then the infinite mass constraint can be avoided [10]. The simplest such media are known as *pentamode* media which represents a limiting class of anisotropic media in which five modes (hence 'penta') of shear combine with one mode of stress and strain. Generalizing further, Norris also showed how utilizing 'pentamode-to-pentamode' transformation media allowed the resultant medium to be inertially isotropic.

Another reason why transformation electromagnetics is so natural is that electromagnetism is *pre-metric* [11]. By contrast acoustics is not pre-metric, and another of our aims in this work was to examine carefully how the metric should be interpreted in any transformational acoustics scheme (cf. Section 6, point 2).

We endevoured in this project to examine the above literature carefully and attempt to formulate transformation acoustics more formally, focusing specifically on purely *spatial* transformations. Several approaches were adopted and described in the sections below. We commence in Section 2 by summarizing linearized acoustics theory from the most general covariant perspective. Then, in Section 3 we derive covariant second order wave equations for the fluid displacement, pressure and stress tensor. In Section 4 we generalize the linearized theory to include anisotropic mass and thereby obtain generalized second order equations for the pressure and the stress. These appear to be quite new. Likewise, we derive the relevant second order equations for pentamode media in Section 5. In Section 6 we return to first order equations that retain the conceptual clarity of the original equations set out in sections 3 and 4, and derive a key theorem, which, for the first time, gives a precise covariant recipe for inertial acoustic cloaking, in which the elasticity tensor takes its simplest form. In Section 7 we extend the theory to pentamodes and show in particular, via another key theorem (Theorem 2 of Section 7.1), how the technical requirement that the tensor defining the pentamode medium (S^{ij}) be divergence free can be manufactured. In Section 8 we apply the developed formalism to a specific and novel example in acoustic cloaking, giving the explicit medium cloaking recipes for pentamode media in the two limits of inertial cloaking and for isotropic mass density. Some commentary of how the formalism might be extended beyond pentamodes is provided in Section 9.

In Section 10 we take a different tack in which we attempt to produce a *geometric* transformation theory, i.e. a method that retains only directions of energy transport (i.e. rays), and then defines how the rate of transport must be modified to produce a given ray deformation. Although such an approach is necessarily imperfect, it did yield a very general recipe (Equation (127)) for transformation media that can be applied across multiple disciplines, including acoustics. An explicit cloaking recipe is discussed.

Finally, in Section 11 we present some analysis in which first order acoustic equations are tied as closely as possible to their electromagnetic counterparts. The formalism is applied specifically to the design of an acoustic carpet cloak.

2 Linearised Acoustics Theory

We here summarize the *linear* theory of elasticity, expressed in an arbitrary spatial coordinate system. This form invariance is essential in discussing transformation acoustics.

Firstly we have the equation of motion for a mass element of the fluid:

$$D_k \sigma_m^{\ k} = \rho g_{mn} \ddot{u}^n \ , \tag{2}$$

(see [13] §2). Here D_k is a spatial *covariant* derivative, $\sigma_m^{\ k}$ is the stress tensor (i.e. $T_m = \sigma_m^{\ k} n_k$ is the stress (= force per unit area) vector acting through a unit area specified by the normal n_k), ρ is the (isotropic) mass density, g_{mn} is the spatial metric (expressed in arbitrary coordinates), and $\ddot{u}$ represents the acceleration of the mass element in question.

Next, we have the *strain tensor* given by

$$e_{nk} \ = \ \frac{1}{2}\left(D_k u_k + D_n u_k\right) = D_{(n} u_{k)} = D_{(n} g_{k)m} u^m \ , \tag{3}$$

(see [13] §3). It represents the deformation of the line element that occurs when the body is deformed from an initial configuration S to a deformed configuration $\bar{S}$, i.e.

$$d\bar{s}^2 - ds^2 = 2e_{nk}dx^n dx^k \ . \tag{4}$$

In the linearized theory we assume that stress and strain are linearly related, i.e.

$$\sigma_i{}^j = C_i{}^{jkl} e_{kl} \ . \tag{5}$$

(see [13] §4). For any linear medium one can decompose the stress tensor as:

$$\begin{aligned} \sigma_i{}^j &= \left(\sigma_i{}^j - \frac{1}{3}\sigma_k{}^k \delta_i{}^j\right) + \frac{1}{3}\sigma_k{}^k \delta_i{}^j \ , & (6)\\ &= (\sigma_i{}^j + p\delta_i{}^j) - p\delta_i{}^j \ , & (7)\\ &= \not{\sigma}_i{}^j - p\delta_i{}^j \ , & (8) \end{aligned}$$

where $p := -\frac{1}{3}\sigma_i{}^i$ and $\not{\sigma}_i{}^j := \sigma_i{}^j - \frac{1}{3}\sigma_k{}^k \delta_i{}^j = \sigma_i{}^j + p\delta_i{}^j$ (see [14] §2). Note that the pressure is related in a trivial way to the trace of the stress, whereas $\not{\sigma}_i{}^j$ is the trace-free part of the stress (i.e. the shear stresses). If a body undergoes *hydrostatic compression* a uniform, isotropic *pressure* p, acts on every unit surface area of the body, directed normally inwards. In that case the shear stresses vanish and $\sigma_i{}^j = -p\delta_i{}^j$.

3 Equations of Motion for Linearized Acoustics

We now proceed to derive equations of motion for a linearized medium. We do this in various ways:

1. by deriving a set of coupled equations for the displacement u^i.
2. by deriving an equation of motion for the (isotropic) pressure p coupled to the stresses
3. by deriving an equation of motion for the stress tensor $\sigma_i{}^j$ coupled to the pressure.

3.1 Equation of Motion for the Displacement

Combining Eq. (2) with Eqs. (5) and (3)

$$D_k C_m{}^{kij} D_i g_{ja} u^a \quad = \quad \rho g_{mn} \ddot{u}^n \ . \tag{9}$$

Now compatibility between the metric and the covariant derivative[2] requires that $D_i g_{jk} = 0$ so that the last equation can be written as

$$D_k C_m^{\ kij} g_{ja} D_i u^a = \rho g_{mn} \ddot{u}^n \ . \tag{10}$$

This equation is three coupled equations for the displacement u^i. Although building a successful transformation acoustics scheme based on Eq. (10) is technically challenging, the compactness and generality of Eq. (10) make this idea very tempting for future investigation.

3.2 Equation of Motion for the Pressure

Combining Eq. (2) with Eq. (8)

$$D_j \not{\sigma}_i^{\ j} - D_i p = \rho g_{ik} \ddot{u}^k \ . \tag{11}$$

Combining $p = -\frac{1}{3}\sigma_i^{\ i}$ with Eq. (5) and Eq. (3) we obtain

$$p = -\frac{1}{3} C_i^{\ ikl} e_{kl} = -\frac{1}{3} C_i^{\ ikl} D_k g_{lm} u^m \ . \tag{12}$$

Now since $C_i^{\ jkl}, D_i$ and g_{ij} are time independent

$$\ddot{p} = -\frac{1}{3} C_i^{\ ikl} D_k g_{lm} \ddot{u}^m \ . \tag{13}$$

Combining Eq. (11) with Eq. (13)

$$\ddot{p} = \frac{1}{3} C_i^{\ ikl} D_k \left[\frac{1}{\rho} D_l p \right] - \frac{1}{3} C_i^{\ ikl} D_k \left[\frac{1}{\rho} D_m \not{\sigma}_l^{\ m} \right] \ . \tag{14}$$

Note the last term couples the equation of motion for pressure to the shear stresses.

3.3 Equation of Motion for the Stress Tensor

Similarly, we can obtain equations of motion for the shear stresses of any linearized medium. Combining Eq. (8) with Eq. (5) and Eq. (3) yields

$$\begin{aligned} \not{\sigma}_i^{\ j} &= C_i^{\ jkl} e_{kl} - \frac{1}{3} C_m^{\ mkl} e_{kl} \delta_i^{\ j} \ , & (15) \\ &= C_i^{\ jkl} D_k g_{ln} u^n - \frac{1}{3} \delta_i^{\ j} \left(C_m^{\ mkl} D_k g_{ln} u^n \right) \ , & (16) \end{aligned}$$

[2] This is how a unique covariant derivative is determined from a metric g_{ij}.

whence

$$\ddot{\phi}_i{}^j = C_i{}^{jkl} D_k g_{ln} \ddot{u}^n - \frac{1}{3}\delta_i{}^j \left(C_m{}^{mkl} D_k g_{ln} \ddot{u}^n \right) \ . \tag{17}$$

Combining the last equation with Eq. (11)

$$\begin{aligned}
\ddot{\phi}_i{}^j &= C_i{}^{jkl} D_k \left[\frac{1}{\rho} D_m \phi_l{}^m - \frac{1}{\rho} D_l p \right] \\
&\quad - \frac{1}{3}\delta_i{}^j C_n{}^{nkl} D_k \left[\frac{1}{\rho} D_m \phi_l{}^m - \frac{1}{\rho} D_l p \right] \\
&= C_i{}^{jkl} D_k \left[\frac{1}{\rho} D_m \phi_l{}^m \right] - \frac{1}{3}\delta_i{}^j C_n{}^{nkl} D_k \left[\frac{1}{\rho} D_m \phi_l{}^m \right] \\
&\quad - C_i{}^{jkl} D_k \left[\frac{1}{\rho} D_l p \right] + \frac{1}{3}\delta_i{}^j C_n{}^{nkl} D_k \left[\frac{1}{\rho} D_l p \right] \ .
\end{aligned} \tag{18}$$

Note that the last two terms couple the equation of motion for the stresses to the pressure.

4 Linearized Acoustics with Anisotropic Mass

It was previously mentioned that the idea of building a transformation acoustics scheme based on Eq. (10) is tempting. However, Eq. (10) is somewhat incomplete, in that it does not include an anisotropic mass. One is thus motivated to ask: what is the equivalent of Eq. (10) with an anisotropic mass? One technical point is how the mass density tensor should be properly indexed. This issue is dealt with in detail in Appendix A, and we here assume the form of the mass density should be written as $\rho^i{}_j$. With anisotropic mass Eq. (2) generalizes to

$$D_k \sigma_m{}^k = g_{mn} \rho^n{}_l \ddot{u}^l \ . \tag{19}$$

We assume that there exists $\left(\rho^{-1}\right)^i{}_j$ such that $\left(\rho^{-1}\right)^i{}_j \rho^j{}_k = \delta^i{}_k$. Omitting algebraic details, the three equations of motion analogous to Eqs. (10), (14) and (18) are straightforwardly derived as

$$D_j \left[C_i{}^{jmn} g_{mp} \right] D_n u^p = g_{ik} \rho^k{}_l \ddot{u}^l \ , \tag{20}$$

$$\begin{aligned}
\ddot{p} &= \frac{1}{3} C_i{}^{ikl} D_k \left[g_{lm} \left(\rho^{-1}\right)^m{}_p g^{pq} \right] D_q p \\
&\quad - \frac{1}{3} C_i{}^{ikl} D_k \left[g_{lm} \left(\rho^{-1}\right)^m{}_p g^{pq} \right] D_r \phi_q{}^r \ ,
\end{aligned} \tag{21}$$

and

$$
\begin{aligned}
\ddot{\not\sigma}_i{}^j &= C_i{}^{jkl} D_k \left[g_{ln} \left(\rho^{-1}\right)^n{}_p g^{pq} \right] D_r \not\sigma_q{}^r \\
&\quad - \frac{1}{3} \delta_i{}^j C_m{}^{mkl} D_k \left[g_{ln} \left(\rho^{-1}\right)^n{}_p g^{pq} \right] D_r \not\sigma_q{}^r \\
&\quad - C_i{}^{jkl} D_k \left[g_{ln} \left(\rho^{-1}\right)^n{}_p g^{pq} \right] D_q p \\
&\quad + \frac{1}{3} \delta_i{}^j C_m{}^{mkl} D_k \left[g_{ln} \left(\rho^{-1}\right)^n{}_p g^{pq} \right] D_q p \ .
\end{aligned} \tag{22}
$$

Relative to the extant literature [9, 13, 14], these equations appear to be new in their use of anisotropic mass.

5 Pentamode Media

A pentamode medium [9] is characterized by the following form for the elasticity tensor

$$
C_i{}^{jkl} = \lambda g_{im} S^{mj} S^{kl} \ , \tag{23}
$$

where S^{ij} is symmetric and invertible. An equation for the pressure in terms of the strain is obtained by combining $p = -\frac{1}{3}\sigma_i{}^i$ with $\sigma_i{}^j = C_i{}^{jkl} e_{kl}$ (Eq. (5)) and $e_{kl} = D_{(k} g_{l)m} u^m$ (Eq. (3)) to obtain

$$
p = -\frac{1}{3} K \left(g_{im} S^{im} \right) \left(g_{pq} S^{qr} D_r u^p \right) \ . \tag{24}
$$

From this we can obtain the shear stress as

$$
\begin{aligned}
\not\sigma_i{}^j &= \sigma_i{}^j + p \delta_i{}^j \ , \\
&= K g_{im} S^{mj} S^{kl} D_k g_{lm} u^m - \frac{1}{3} K \left(g_{im} S^{im} \right) \left(g_{pq} S^{qr} D_r u^p \right) \delta_i{}^j \ , \\
&= K \left(g_{pq} S^{qr} D_r u^p \right) \left[g_{ik} S^{kj} - \frac{1}{3} \left(g_{mn} S^{mn} \right) \delta_i{}^j \right] \ .
\end{aligned} \tag{25}
$$

We now introduce the so-called ‘pseudo-pressure’ [9]

$$
\tilde{p} = -K \left(g_{pq} S^{qr} D_r u^p \right) \ , \tag{26}
$$

and proceed to express p, $\not\sigma_i{}^j$ and $\sigma_i{}^j$ in terms of $\tilde{p}$ as

$$
p = \frac{1}{3} \tilde{p} \left(g_{mn} S^{mn} \right) \ , \tag{27}
$$

$$
\not\sigma_i{}^j = -\tilde{p} \left[g_{ik} S^{kj} - \frac{1}{3} \left(g_{mn} S^{mn} \right) \delta_i{}^j \right] \ , \tag{28}
$$

$$
\sigma_i{}^j = -\tilde{p} \left[g_{ik} S^{kj} \right] \ . \tag{29}
$$

On the right hand sides of Eqs. (27), (28) and (27), the only time dependent quantity is the pseudo-pressure. Accordingly, the pressure, shear stresses and stress tensor consist of a time-independent geometric structure modulated by a time-dependent scalar, namely the pseudo-pressure. This observation is related to the fact that pentamodes are well-suited to transformation acoustics. Now, combining Eqs. (19) and (29) we obtain straightforwardly

$$D_j \left[-\tilde{p} g_{im} S^{mj} \right] = g_{ik} \rho^k{}_l \ddot{u}^l \ . \tag{30}$$

After multiplying through by $\left(\rho^{-1}\right)^p{}_q g^{qi}$ and utilizing the fact that the covariant derivative commutes with the metric we obtain

$$\ddot{u}^p = -\left(\rho^{-1}\right)^p{}_m D_n \left[\tilde{p} S^{mn} \right] \ . \tag{31}$$

Differentiating Eq. (26) twice with respect to time and substituting for $\ddot{u}^p$ from Eq. (31) yields

$$\ddot{\tilde{p}} = K g_{pq} S^{qr} D_r \left(\rho^{-1}\right)^p{}_m D_n \left[\tilde{p} S^{mn} \right] \ . \tag{32}$$

If S^{ij} is assumed to be divergence free, i.e. $D_i S^{ij} = 0$ then the preceeding equation is straightforwardly manipulated to

$$\ddot{\tilde{p}} = K D_i \left[S^{ij} g_{jk} \left(\rho^{-1}\right)^k{}_l S^{lm} \right] D_m \tilde{p} \ . \tag{33}$$

6 Inertial Transformation Acoustics

In this section we aim to provide a general recipe for so-called 'inertial transformation acoustics', i.e. a transformational scheme that maps a shear-free medium with anisotropic inertial mass to another similar medium. Therefore the only constitutive parameters altered by the transformation scheme are the bulk modulus (λ), and the (anisotropic) mass tensor $\rho^i{}_j$. In contrast to other authors ([6], [8], [9]) who derive transformation algorithms for second order wave equations, in this section we retain first order equations. This choice provides better conceptual clarity, and keeps the methodology closer to transformation optics. There are two requirements imposed to achieve the recipe for transformation acoustics:

1. a simple conversion from covariant to ordinary derivatives.
2. we need to have (effective) constitutive quantities where the metric can be hidden after it has been transformed.

We start with the force density f_i given by

$$f_i = g_{ij} \rho^j{}_k \ddot{u}^k \ , \tag{34}$$

and the constitutive laws

$$C_i{}^{jkl} = \lambda \delta_i{}^j g^{kl} \ , \tag{35}$$

$$C_i{}^{jkl} e_{kl} = \sigma_i{}^j \ , \tag{36}$$

and finally the differential equations

$$D_i \sigma_j{}^i = f_j \ , \tag{37}$$

$$e_{ij} = \frac{1}{2} \left(D_i g_{jk} u^k + D_j g_{ik} u^k \right) \ , \tag{38}$$

but avoid in the following using second covariant derivatives.

Substituting Eq. (36) into (37)

$$D_i \left[C_j{}^{ikl} e_{kl} \right] = f_j \ , \tag{39}$$

and then using Eq. (35) we obtain

$$D_i \left[\lambda \delta_j{}^i \left(g^{kl} e_{kl} \right) \right] = f_j \ . \tag{40}$$

Finally substituting Eq. (34) into the above equation yields

$$D_i \left[\lambda \left(g^{pq} e_{pq} \right) \right] = g_{ij} \rho^j{}_k \ddot{u}^k \ . \tag{41}$$

Now the strain equation, Eq. (38) can be decomposed into a traceless part and an isotropic part:

$$\begin{aligned} e_{ij} &= g_{im} \left[\not{e}^m_j + \left(g^{pq} e_{pq} \right) \delta^m_j \right] \ , & (42) \\ &= g_{im} \not{e}^m_j + \left(g^{pq} e_{pq} \right) g_{ij} \ . & (43) \end{aligned}$$

This decomposition of the strain tensor is quite crucial, since it facilitates, as we show subsequently, a simple conversion from covariant to ordinary derivatives.[3] Note that the traceless part, $\not{e}^i{}_j$ does not couple with Eqs. (34), (35), (36) and (37). Furthermore,

$$g^{pq} e_{pq} = g^{pq} \frac{1}{2} \left(D_p g_{qm} u^m + D_q g_{pr} u^r \right) = D_r u^r \ , \tag{44}$$

[3]This decomposition of e_{ij} is *irreducible* under $\mathrm{GL}(4, \mathbb{R})$.

the last equality arising because of the covariant compatibility of the metric. Using the above, we express the differential equation Eq. (41) and the strain equation Eq. (38) in the following way

$$D_i \left(\lambda \xi \right) = g_{ij} \rho^j{}_k \ddot{u}^k \ , \tag{45}$$

$$\xi = D_i u^i \ . \tag{46}$$

Eqs. (45) and (46) constitute four equations in four unknowns. Using well known properties of the covariant derivative, these last two equations can be written in terms of ordinary derivatives as

$$\partial_i \left(\lambda \xi \right) = g_{ij} \rho^j{}_k \ddot{u}^k \ , \tag{47}$$

$$\xi = \frac{1}{\sqrt{\det g_{ij}}} \frac{\partial}{\partial x^p} \left[\sqrt{\det g_{ij}} u^p \right] \ , \tag{48}$$

which again is 4 equations in four unknowns (i.e. three unknown u^α and one unknown ξ). We are now ready to state the basic "inertial cloaking" version of transformation acoustics (cf. [9]). We do this via a theorem.

6.1 Basic Transformation Acoustics Theorem

Theorem: If u^p and ξ are solutions of Eqs. (47) and (48), then

$$\xi' = \frac{\sqrt{\det \bar{g}_{ij}}}{\sqrt{\det g'_{ij}}} \bar{\xi} \ , \tag{49}$$

and

$$u'^p = \frac{\sqrt{\det \bar{g}_{ij}}}{\sqrt{\det g'_{ij}}} \bar{u}^p \ , \tag{50}$$

are solutions of

$$\partial_i \left(\lambda' \xi' \right) = g'_{ij} \rho'^j{}_k \ddot{u}'^k \ , \tag{51}$$

$$\xi' = \frac{1}{\sqrt{\det g'_{ij}}} \frac{\partial}{\partial x^p} \left(\sqrt{\det g'_{ij}} u'^p \right) \ , \tag{52}$$

where

$$\lambda' = \frac{\sqrt{\det g'_{ij}}}{\sqrt{\det \bar{g}_{ij}}} \bar{\lambda} \ , \tag{53}$$

and

$$\rho'^{i}{}_{j} = \frac{\sqrt{\det g'_{ij}}}{\sqrt{\det \bar{g}_{ij}}} g'^{im} \bar{g}_{mn} \bar{\rho}^{n}{}_{j} \ . \tag{54}$$

Note the following definitions (see Appendix B):

$$\bar{\xi} = \varphi_* \xi \ , \bar{u}^i = (\varphi_*)^i{}_j \, u^j \ , \bar{\lambda} = (\varphi_* \lambda) \ , \bar{\rho}^m_{\ n} = (\varphi_* \rho)^m_{\ \ n} \ , \bar{g}_{ij} = (\varphi_*)^m_{\ i} (\varphi_*)^n_{\ j} g_{mn} \ , \tag{55}$$

where g'_{ij} is a spatial metric. Stating the theorem in words: under a transformation solutions of the elastodynamic equations are mapped to solutions of the elastodynamic equations at the price of redefining the constitutive parameters. Note how the metric is 'hidden' in Eq. (54) after transformation. The proof of the theorem is given in Appendix C.

Lemma: $p' = -\frac{1}{3}\sigma'_i{}^i$ obeys $p' = \bar{p}$.

Proof: $p' = -\frac{1}{3}\lambda' \delta_i{}^i g'^{kl} e'_{kl} = -\frac{3}{3}\lambda' \xi' = -\lambda' \xi' = -\bar{\lambda}\bar{\xi} = \bar{p}$.

7 Pentamode Acoustic Transformation

The previous section presented the basic transformation acoustics theorem for a trivial medium in which the elasticity tensor took its simplest form as given by Eq. (35). Here we extend the theorem to pentamode media where the elasticity tensor is given by Eq. (23). Note that since the trivial medium $C^{ijkl} = \lambda g^{ij} g^{kl}$ (which is equivalent to Eq. (35)) is also a pentamode, the theory of this section encompasses the transformation (trivial medium) $\rightarrow$ (pentamode medium) as a sub-case. For clarity we restate all the starting equations for linearised elasticity (i.e. small displacement) in a pentamode material as

$$\begin{aligned} C_i{}^{jkl} &= \lambda g_{im} S^{mj} S^{kl} \ , && (56)\\ \sigma_i{}^j &= C_i{}^{jkl} e_{kl} \ , && (57)\\ f_i &= g_{ij} \rho^j{}_k \ddot{u}^k \ , && (58)\\ D_j \sigma_i{}^j &= f_i \ , && (59)\\ e_{ij} &= \frac{1}{2}\left(D_i g_{jk} u^k + D_j g_{ik} u^k\right) \ . && (60) \end{aligned}$$

To these we add a crucial additional assumption, namely that

$$D_i S^{ij} = 0 \ . \tag{61}$$

It is emphasised again, that the subsequent development is restricted to first order equations, and no attempt is made to generate wave equations that will involve second spatial (covariant) derivatives. Combining Eqs. (56)-(59) yields

$$D_j \left[\lambda g_{im} S^{mj} \left(S^{kl} e_{kl} \right) \right] = g_{ip} \rho^p{}_q \ddot{u}^q \ . \tag{62}$$

The strain of Eq. (60) can be decomposed with respect to S^{ij} as

$$e_{ij} = S_{im} \left[\not{e}^m{}_j + \left(S^{pq} e_{pq} \right) \delta^m_j \right] = S_{im} \not{e}^m{}_j + \left(S^{pq} e_{pq} \right) S_{ij} \ , \tag{63}$$

where we note that the trace-free part of the strain $\not{e}^i{}_j$, does not couple to Eq. (57). Note, in Eq. (63) (and subsequently) the symbol S_{im} refers to the im^{th} component of the *inverse* of S^{jk}. For a pentamode medium only the component of the strain given by $\xi := S^{ij} e_{ij}$ couples to Eq. (57). The decomposition given by Eq. (63) is actually quite crucial as it permits us to develop equations in which the covariant derivative is easily related to the partial derivatives. In turn, this allows transformation acoustics to be performed successfully. Now, by using Eq. (60), we can easily show by relabelling indices and the symmetry of S^{pq}, that the term in parentheses in the last equation can be written as

$$S^{pq} e_{pq} = S^{pq} D_p \left(g_{qr} u^r \right) \ . \tag{64}$$

Thus, in a manner analogous to Eqs. (45) and (46) we arrive at

$$D_j \left[\lambda \xi g_{im} S^{mj} \right] = g_{ip} \rho^p{}_q \ddot{u}^q \ , \tag{65}$$

$$\xi := S^{ij} e_{ij} = S^{ij} D_i \left(g_{jk} u^i \right) \ , \tag{66}$$

which is four equations in four unknowns. If we now make use of the additional assumption Eq. (61), and the invertibility of S^{ij}, then the above two equations become

$$D_i \left(\lambda \xi \right) = S_{ij} \rho^j{}_k \ddot{u}^k \ , \tag{67}$$

$$\xi = D_i \left(S^{ij} g_{jk} u^k \right) \ , \tag{68}$$

or, with further use of the invertibility of S^{ij}

$$D_i \left(\lambda \xi \right) = S_{ij} \rho^j{}_k g^{kl} S_{lm} \ddot{w}^m \ , \tag{69}$$

$$\xi = D_i w^i \ , \tag{70}$$

where $w^i := S^{ij} g_{jk} u^k$. These two sets of first-order equations encode the dynamics of pentamode materials in two slightly distinct ways. As before, both sets (i.e. Eqs. (67)-(68) and (69) and (70)) may be expressed in terms of ordinary derivatives:

$$\partial_i (\lambda\xi) = S_{ij} \rho^j{}_k \ddot{u}^k \ , \tag{71}$$

$$\xi = \frac{1}{\sqrt{\det g_{pq}}} \partial_i \left[\sqrt{\det g_{pq}} S^{ij} g_{jk} u^k \right] \ , \tag{72}$$

and

$$\partial_i (\lambda\xi) = S_{ij} \rho^j{}_k g^{kl} S_{lm} \ddot{w}^m \ , \tag{73}$$

$$\xi = \frac{1}{\sqrt{\det g_{pq}}} \partial_i \left[\sqrt{\det g_{pq}} w^i \right] \ . \tag{74}$$

Before stating the relevant pentamode transformation acoustics theorems, we note that the assumption of vanishing divergence of S^{ij} (i.e. $D_i S^{ij} = 0$) must hold true for the tensor S^{ij} both before *and* after transformation acoustics is applied. This leads to an additional constraint that in practice may be quite hard to satisfy, particularly if the transformational scheme is applied to Eqs. (71) and (72). By contrast, the set (73) and (74) has additional flexibility in the choice of the transformed tensor S'_{ij} which can be exploited to manufacture $D_i S'^{ij} = 0$. Hence in the next section we give two transformational acoustic theorems, one for set (71) and (72) and one for set (73) and (74).

Another point concerns the quantity ξ. It is related to Norris' pseudo pressure, $\tilde{p}$ [9] via $\tilde{p} = -\lambda\xi$, and can be called the 'trace' of the strain with respect to S^{ij}.

7.1 Pentamode-to-Pentamode Transformation Acoustics Theorems

We now state the transformation acoustics theorem analogous to that given in Section 6.1:

Theorem 1

If ξ and u^k are solutions of Eqs. (71) and (72), then

$$\xi' = \frac{\sqrt{\det \bar{g}_{pq}}}{\sqrt{\det g'_{pq}}} \bar{\xi} \quad \text{and} \quad u'^i = \frac{\sqrt{\det \bar{g}_{pq}}}{\sqrt{\det g'_{pq}}} \bar{u}^i \tag{75}$$

are solutions of

$$\partial_i \left(\lambda' \xi'\right) = S'_{ij} \rho'^{j}{}_{k} \ddot{u}'^{k} \ , \tag{76}$$

$$\xi' = \frac{1}{\sqrt{\det g'_{pq}}} \partial_i \left[\sqrt{\det g'_{pq}} S'^{ij} g'_{jk} u'^{k} \right] \ , \tag{77}$$

where

$$S'^{ij} := \bar{S}^{im} \bar{g}_{mn} g'^{nj} \ , \tag{78}$$

$$\rho'^{j}{}_{k} := \frac{\sqrt{\det g'_{pq}}}{\sqrt{\det \bar{g}_{pq}}} S'^{jm} \bar{S}_{mn} \bar{\rho}^{n}{}_{k} \ , \tag{79}$$

$$\lambda' := \frac{\sqrt{\det g'_{pq}}}{\sqrt{\det \bar{g}_{pq}}} \bar{\lambda} \ . \tag{80}$$

Here (see Appendix B for notation) $\bar{\xi} = \varphi_* \xi$ and $\bar{u}^i = (\varphi_* u)^i$ are transformed variables; $\bar{S}^{ij} = (\varphi_* S)^{ij}$, $\bar{\rho}^{i}{}_{j} = (\varphi_* \rho)^{i}{}_{j}$ and $\bar{\lambda} = \varphi_* \lambda$ are transformed constitutive parameters; $\bar{g}^{ij} = (\varphi_* g)^{ij}$ is the transformed metric and g'^{ij} is some other metric.

To give the theorem in words, the primed quantities are quantities obtained after:

transformation + reinterpretation as a new medium

The proof of the theorem is entirely analogous to that given in Appendix C. Attention is also drawn in particular to Eq. (78) which is effectively a requirement on the new S'^{ij}. In order for a transformational scheme based on Eqs. (75)-(80) to work it is also necessary for $D_i S'^{ij} = 0$, i.e. the primed S^{ij} must be covariantly conserved. Satisfying this requirement turns out to be quite problematic. However, this difficulty may be circumvented by instead basing the transformational acoustics scheme on the equation set (73)-(74). We now give the second transformational acoustic theorem based on the equation set (73) and (74).

Theorem 2

If ξ and w^k are solutions of Eqs. (73) and (74), then

$$\xi' = \frac{\sqrt{\det \bar{g}_{pq}}}{\sqrt{\det g'_{pq}}} \ \bar{\xi} \quad \text{and} \quad w'^{i} = \frac{\sqrt{\det \bar{g}_{pq}}}{\sqrt{\det g'_{pq}}} \ \bar{w}^{i} \tag{81}$$

are solutions of

$$\partial_i \left(\lambda' \xi'\right) = S'_{ij} \rho'^{j}{}_{k} g'^{kl} S'_{lm} \ddot{w}'^{m} \ , \tag{82}$$

$$\xi' = \frac{1}{\sqrt{\det g'_{pq}}} \partial_i \left[\sqrt{\det g'_{pq}} w'^{i} \right] \ , \tag{83}$$

provided

$$S'^{ij} \rho'^{j}{}_{k} g'^{kl} S'_{lm} = \frac{\sqrt{\det g'_{pq}}}{\sqrt{\det \bar{g}_{pq}}} \ \bar{S}_{ia} \bar{\rho}^{a}{}_{b} \bar{g}^{bc} \bar{S}_{cm} \ , \tag{84}$$

and

$$\lambda' = \frac{\sqrt{\det g'_{pq}}}{\sqrt{\det \bar{g}_{pq}}} \ \bar{\lambda} \ . \tag{85}$$

Here quantities denoted with an overbar are the transformed quantities, whereas those denoted with a prime are quantities actually implemented in the final medium. The quantity g'_{pq} is an arbitrary metric, usually set to be equal to g_{pq}, the metric of Euclidean space.

Note that if $S^{ij} = g^{ij}$ and $S'^{ij} = \bar{S}^{ij}$, the above transformation scheme replicates that of Norris [9], taking a simple shear-free material as being a special case of a pentamode. Our development, in contrast to Norris, has been consistently first order.

When we consider a specific application of transformational acoustics in Section 8, we will exploit the additional flexibility referred to above and use Theorem 2. Since exploitation of Theorem 2 is more fruitful than Theorem 1, we give a detailed proof in Appendix D.

8 Application: Cylindrical Acoustic Cloak

We now give a specific application of the above formalism to acoustic cloaking, illustrating the use of the second theorem given in the last section. We take the radial transformation to be

$$\begin{aligned} \tilde{r} &= \left(r^2 + a^2\right)^{1/2} \ , \\ \tilde{\theta} &= \theta \ . \end{aligned} \tag{86}$$

The origin point $r = 0$ is mapped to the circle $\tilde{r} = a$, and for large r we have $\tilde{r} \approx r$. Note that this improves on the ‘standard’ cloaking transformation

$$\begin{aligned} \tilde{r} &= \left(\frac{b-a}{b}\right) r + a \ , \\ \tilde{\theta} &= \theta \ . \end{aligned} \tag{87}$$

which, although extensively used in the literature right from the inception of transformation optics [1], fails to satisfy $\tilde{r} \to r$ as $r \to \infty$.

In order to proceed we need to first express the underlying Euclidean metric $g_{ij} = \delta_{ij}$ in cylindrical polar coordinates

$$r = \left(x^2 + y^2 + z^2\right)^{1/2} \;\; , \;\; \theta = \tan^{-1}\left(\frac{y}{x}\right) \; , \tag{88}$$

with the z-coordinate unaltered. Using $g_{i'j'} = \frac{\partial x_i}{\partial x_{i'}} \frac{\partial x_j}{\partial x_{j'}} g_{ij}$, we find the polar representation $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{zz} = 1$, i.e. $g_{i'j'} = \mathrm{diag}\left(1, r^2, 1\right)$. The inverse is given by $g^{i'j'} = \mathrm{diag}\left(1, r^{-2}, 1\right)$. For convenience we now drop the primes so that g_{ij} represents the untransformed Euclidean metric expressed in cylindrical coordinates.

In order to implement the transformation scheme we need to calculate the push-forward quantities $(\varphi_* g^{-1})^{ij}$, $(\varphi_* g)_{ij}$, and $(\varphi_* \lambda)$, where φ is the mapping given by Eqs. (86), and λ is the bulk modulus of the original homogeneous medium denoted by $\lambda = \kappa$, where κ is a constant. We find

$$(\varphi_* g^{-1})^{ij} = \mathrm{diag}\left[(\tilde{r}^2 - a^2)/\tilde{r}^2, (\tilde{r}^2 - a^2)^{-1}, 1\right] \; , \tag{89}$$

$$(\varphi_* g)_{ij} = \mathrm{diag}\left[\tilde{r}^2/(\tilde{r}^2 - a^2), (\tilde{r}^2 - a^2), 1\right] \; , \tag{90}$$

$$(\varphi_* \lambda) = \kappa \; . \tag{91}$$

We are now ready to apply the transformational scheme of Theorem 2. We make the following assumptions concerning the initial (i.e. untransformed) medium:

1. The initial medium is a *simple* fluid, i.e.

$$C_i^{jkl} = \lambda \delta_i^{\;j} g^{kl} \; . \tag{92}$$

2. The initial medium is characterized by an *isotropic* mass density, i.e.

$$\rho^i{}_j = \rho \delta^i{}_j \; . \tag{93}$$

3. The initial medium is characterized by a *constant* bulk modulus, i.e.

$$\lambda = \kappa \; . \tag{94}$$

4. The initial medium is characterized by a *constant* mass density, i.e.

$$\rho = M = \text{ constant} \; . \tag{95}$$

After application of Theorem 2 we find that Eq. (84) becomes

$$S'_{ij}\rho'^{j}{}_{k}g'^{kl}S'_{lm} = \frac{\sqrt{\det g'_{pq}}}{\sqrt{\det \bar{g}_{pq}}}\ M\bar{g}_{im}\ , \tag{96}$$

$$\lambda' = \frac{\sqrt{\det g'_{pq}}}{\sqrt{\det \bar{g}_{pq}}}\ \bar{\lambda}\ , \tag{97}$$

or, explicitly for the transformation of Eqs. (86) (now dropping the tilde),

$$S'_{ij}\rho'^{j}{}_{k}g'^{kl}S'_{lm} = \\ \frac{\sqrt{\det g'_{pq}}}{\sqrt{\det g_{pq}}}\ M\left[\mathrm{diag}\left(\frac{\mathrm{r}^2}{(\mathrm{r}^2-\mathrm{a}^2)^{1/2}}, \frac{(\mathrm{r}^2-\mathrm{a}^2)^{3/2}}{\mathrm{r}}, \frac{(\mathrm{r}^2-\mathrm{a}^2)^{1/2}}{\mathrm{r}}\right)\right]\ , \tag{98}$$

$$\lambda' = \frac{\sqrt{\det g'_{pq}}}{\sqrt{\det g_{pq}}}\ \frac{(r^2-a^2)^{1/2}}{r}\ \kappa\ . \tag{99}$$

We now focus on the possibilities for the medium designed to mimic the transformation. One natural assumption is that the medium exists, like its untransformed antecedent, in Euclidean space, so that $g'_{ij} = g_{ij} = \mathrm{diag}(1, r^2, 1)$ and $g'^{ij} = g^{ij} = \mathrm{diag}(1, r^{-2}, 1)$. Eqs. (98) and (99) then become

$$S'_{ij}\rho'^{j}{}_{k}g^{kl}S'_{lm} = M\left[\mathrm{diag}\left(\frac{\mathrm{r}}{(\mathrm{r}^2-\mathrm{a}^2)^{1/2}}, \frac{(\mathrm{r}^2-\mathrm{a}^2)^{3/2}}{\mathrm{r}}, \frac{(\mathrm{r}^2-\mathrm{a}^2)^{1/2}}{\mathrm{r}}\right)\right]\ , \tag{100}$$

$$\lambda' = \frac{(r^2-a^2)^{1/2}}{r}\ \kappa\ . \tag{101}$$

Evidently, there are various possibilities for satisfying Eq. (100). We consider just two.

8.1 Inertial Cloaking Limit

In this case

$$S'^{ij} = g^{ij} = \mathrm{diag}(1, r^{-2}, 1)\ , \tag{102}$$

and we have for $r > a$

$$\rho'^{i}{}_{j} = M\left[\mathrm{diag}\left(\frac{\mathrm{r}}{(\mathrm{r}^2-\mathrm{a}^2)^{1/2}}, \frac{(\mathrm{r}^2-\mathrm{a}^2)^{3/2}}{\mathrm{r}^3}, \frac{(\mathrm{r}^2-\mathrm{a}^2)^{1/2}}{\mathrm{r}}\right)\right]\ , \tag{103}$$

$$\lambda' = \frac{(r^2 - a^2)^{1/2}}{r} \quad \kappa \ . \tag{104}$$

In this case the requirement that $D_i S'^{ij} = 0$ is automatically satisfied by the covariant compatibility condition $D_i g^{jk} = 0$ which associates the metric g_{ij} with its unique covariant derivative D_i.

8.2 Isotropic Mass Density Limit

In this case we set $\rho'^i{}_j = M' \delta^i{}_j$ and obtain for $r > a$

$$S'_{ij} = \left(\frac{M}{M'}\right)^{1/2} \left[\mathrm{diag} \left(\frac{\mathrm{r}^{1/2}}{(\mathrm{r}^2 - \mathrm{a}^2)^{1/4}}, \mathrm{r}^{1/2}(\mathrm{r}^2 - \mathrm{a}^2)^{3/4}, \frac{(\mathrm{r}^2 - \mathrm{a}^2)^{1/4}}{\mathrm{r}^{1/2}} \right) \right] \ , \tag{105}$$

$$\lambda' = \frac{(r^2 - a^2)^{1/2}}{r} \quad \kappa \ . \tag{106}$$

Now the requirement $D_i S^{ij} = 0$ sets a condition on the radial mass density $M'(r)$. Explicitly, S^{ij} must satisfy

$$\partial_i S^{ij} + \Gamma^i{}_{ai} S^{aj} + \Gamma^j{}_{ai} S^{ai} = 0 \ , \tag{107}$$

where the only non-zero connection coefficients in cylindrical polar coordinates are $\Gamma^r{}_{\theta\theta} = -r$ and $\Gamma^\theta{}_{r\theta} = r^{-1}$. After inverting S_{ij} in Eq. (105) and substituting in the above equation, we find that $M'(r)$ must satisfy

$$\begin{aligned} &\frac{\left(r^2 - a^2\right)^{1/4}}{r^{1/2}} \frac{\partial}{\partial r} \left[\ln \left(\frac{M'}{M} \right) \right] + \frac{\partial}{\partial r} \frac{\left(r^2 - a^2\right)^{1/4}}{r^{1/2}} \\ &\qquad + \frac{\left(r^2 - a^2\right)^{1/4}}{r^{3/2}} - \frac{r^{1/2}}{\left(r^2 - a^2\right)^{3/4}} = 0 \ . \end{aligned} \tag{108}$$

The solution is

$$\frac{M'}{M} = \mathrm{const.} \left(1 - \frac{a^2}{r^2}\right)^{1/2} \ , \tag{109}$$

where const. is an integration constant. This mass distribution, together with the bulk modulus distribution (which for this case is still given by Eq. (104)) ensures that acoustic disturbances are bent around the region $r < a$, and, being mass-isotropic, should be easier to achieve than the previous inertial limit case.

9 Transformation acoustics beyond pentamodes

As explained in Section 6, to achieve a valid transformation acoustics scheme it is essential that the equations of motion can be easily cast in terms of ordinary derivatives, as opposed to covariant ones. When the transformation acoustics scheme takes its moves from a pentamode material (Section 7), the effortless conversion from covariant to ordinary derivatives is made possible by the identification of a crucial *scalar*, namely the pseudo pressure $\tilde{p}$. As an alternative, one can make use of the parameter ξ, given that this quantity is a proxy for $\tilde{p}$. For more general materials, the determination of a *single* scalar, such as the pseudo pressure, becomes insufficient so as to characterise the behaviour of sound waves. In particular, shear effects may occur that are truly tensorial in nature. An important consequence follows: the fundamental equations for more general materials do not manifest a simple conversion from covariant to ordinary derivatives, unless suitably manipulated. Ergo, developing a transformation acoustics scheme that goes "beyond pentamodes" is a significant challenge. One solution may come from the identification of *multiple* scalars, describing the numerous features of sound propagation in complex media. To this end, a decomposition of the stress-strain tensor C^{ijkl} put forward in the article [15] by Milton and Cherkaev is employed:

$$C^{ijkl} = \sum_{I=1}^{6} \lambda_I \, S_I^{ij} S_I^{kl} = \lambda_1 \, S_1^{ij} S_1^{kl} + \cdots + \lambda_6 \, S_6^{ij} S_6^{kl}. \tag{110}$$

The scalars λ_I may be regarded as bulk moduli. In addition, the tensors S_I^{ij} are not necessarily full-rank. A pentamode material is defined by the property that five of the parameters λ_I vanish. Contracting C^{ijkl} with the strain e_{ij} allows one to formulate the stress tensor, with one index raised, as

$$\sigma^{ij} = -\sum_{I=1}^{6} \tilde{p}_I S_I^{ij} = -\left(\tilde{p}_1 S_1^{ij} + \cdots + \tilde{p}_6 S_6^{ij}\right). \tag{111}$$

Each of the scalars $\left(\tilde{p}_1, \ldots, \tilde{p}_6\right) := \left(-\lambda_1 S_1^{ij} e_{ij}, \ldots, -\lambda_6 S_6^{ij} e_{ij}\right)$ can be interpreted as a pseudo pressure. As anticipated, the identification of these *multiple* parameters is the key for constructing a transformation acoustics scheme that is applicable beyond pentamodes. In fact, subject to the condition that

$$D_k S_I^{kl} = 0 \quad \text{for all } I, \tag{112}$$

it is possible to identify six coupled wave equations, governing the evolution of the scalars p_I. Notably, the covariant derivatives appearing in such equations of motion are easily converted into ordinary ones. As a result, one can devise a transformation acoustics scheme for media of very general nature. However, it should be recognised that the constrain of Eq. (112), which must be imposed on all S_I, is very restrictive. Thus the scheme outlined above can only be claimed to a be a first step towards the Transformation Acoustics of a non-scalar wave equation. Second, numerous transformation acoustics endeavors that are otherwise productive fail to comply with the requirement (112) for all I.

10 Geometric ('Ray') Transformation Acoustics

In this section we formulate a transformation theory that operates at the level of *geometrical* acoustics, i.e. the goal of an exact field transformation theory (cf. foregoing sections) is compromised in favour of a ray-based theory that seeks to define material properties that can morph *rays* in a prescribed or designed way. The key idea is to regard the rays as geodesics with respect to a metric. A 'speed' is defined at each point on a given geodesic according to the magnitude of the tangent vector at that point. A desired morphing of the rays will be shown to relate to a single parameter $n(\mathbf{r})$, given by the ratio of the speed along the unperturbed geodesics, to the speed along the perturbed geodesics. Such a refractive index can then be related to simple medium parameters such as the contrast in bulk modulus and mass density. This transformational strategy appears to be quite generic. *Any* ray theory (e.g. geometrical optics, geometrical acoustics, etc.) admits a parameter equivalent to the refractive index of geometrical optics, so that relating spatial deformations to the required index distribution (and thereby material parameters) gives us access to e.g. an acoustic cloak or black hole, a refractive-index based cloak or concentrator for optics, etc. One just has to prescribe the desired deformation, and the following theory will then deliver the required index distribution. The price of such generality is that any such transformation device is necessarily imperfect, since the very index contrast used to create the device induces scattering.

10.1 Geodesic Equation

A covariant derivative provides a mechanism for parallel transportation, allowing vectors at different locations to be compared. Insisting that a given vector v, be parallel transported with respect to itself results in the geodesic equation:

$$v^i D_i v^j = v^i \partial_i v^j + \Gamma^j{}_{ki} v^k v^i = 0 \ . \tag{113}$$

Alternatively, if time t affinely parameterizes the curve for which $v = d/dt = v^i \partial_i$ is tangent, then $v^i = dx^i/dt$, and

$$\frac{d^2 x^i}{dt^2} + \Gamma^j{}_{ki} \frac{dx^k}{dt} \frac{dx^i}{dt} = 0 \ . \tag{114}$$

A standard result from differential geometry is that a metric induces a unique covariant derivative for which the connection coefficients are given by

$$\bar{\Gamma}^i{}_{jk} = \frac{1}{2} g^{im} \left(\partial_k g_{jm} + \partial_j g_{km} - \partial_m g_{jk} \right) \ . \tag{115}$$

10.2 Conformal Metric Transformation

The cartesian components of the Euclidean metric of flat three-dimensional space are just δ_{ij}, so that according to Eq. (115) all connection coefficients vanish, leaving

$$\frac{d^2 x^i}{dt^2} = 0 \ . \tag{116}$$

This integrates up to

$$x^j(t) = x^j(0) + v_0^j t \ , \tag{117}$$

i.e. the familiar straight lines of Euclidean geometry. Also, according to Newtonian mechanics, Eq. (117) is the trajectory of a free particle with velocity v_0^j. We now take the Euclidean metric and conformally transform it according to (in Cartesian coordinates)

$$g_{ij} = n^2 \delta_{ij} \ . \tag{118}$$

The transformed metric generates a new covariant derivative for which the connection coefficients are determined from Eq. (115). In Cartesian coordinates we have

$$\bar{\Gamma}^i{}_{jk} = \frac{1}{2} \delta^{im} \left(\delta_{mj} \partial_k + \delta_{mk} \partial_i - \delta_{jk} \partial_m \right) [\ln n] \ . \tag{119}$$

The new covariant derivative produces new geodesics that satisfy

$$\frac{d^2 \bar{x}^i}{dt^2} + \bar{\Gamma}^j{}_{ki} \frac{d\bar{x}^k}{dt} \frac{d\bar{x}^i}{dt} = 0 \ . \tag{120}$$

Equation (119) tells us how the new geodesics arise from the old. The first two terms on the right hand side of Eq. (119) correspond to a reparameterization of the original geodesics from time t to a dilated time, $T(t)$. This is seen by transforming Eq. (113) with $\Gamma^{i}{}_{jk} = 0$ to one in which a 'bad' clock $T(t)$ is used:

$$\frac{d^2x^i}{dt^2} = \dot{T}\frac{d}{dT}\left(\dot{T}\frac{dx^i}{dT}\right) = \dot{T}\left[\dot{T}\frac{d^2x^i}{dT^2} + \left(\partial_j\dot{T}\right)\frac{dx^i}{dT}\frac{dx^j}{dT}\right] = 0 \ , \tag{121}$$

or

$$\frac{d^2x^i}{dT^2} + \partial_j\left(\ln\dot{T}\right)\frac{dx^i}{dT}\frac{dx^j}{dT} = 0 \ , \tag{122}$$

or

$$\frac{d^2x^i}{dT^2} + \frac{1}{2}\delta^{im}\left(\delta_{mj}\partial_k + \delta_{mk}\partial_i\right)\left(\ln\dot{T}\right)\frac{dx^j}{dT}\frac{dx^k}{dT} = 0 \ . \tag{123}$$

Thus we see that indeed the time reparameterization generates the first two terms on the right of Eq. (119) provided $\dot{T} = n$.

The final term of Eq. (119) is the most interesting. It corresponds to *bending* the original geodesic, when there is a *gradient* of $n = \dot{T}$ as we move between neighbouring geodesics. When $\partial_x n = 0$, for example, geodesics parallel to the x-axis in the Euclidean plane deform to ones satisfying

$$\frac{d^2x}{dt^2} + 2\left[\partial_y\left(\ln n\right)\right]\frac{dx}{dt}\frac{dy}{dt} = 0 \ , \tag{124}$$

$$\frac{d^2y}{dt^2} + \left[\partial_y\left(\ln n\right)\right]\left[\left(\frac{dy}{dt}\right)^2 - \left(\frac{dx}{dt}\right)^2\right] = 0 \ , \tag{125}$$

Clearly, the solutions to Eqs. (124)-(125) will describe a geodesic that deviates and curves away from the x-axis as illustrated in Fig. 1 We thus conclude that conformal transformation of the spatial metric has two effects: to reparameterise the geodesics so that n can be interpreted as a refractive index, and to deform the geodesics in a potentially useful way.

10.3 Direct Transformation of Geodesics

The previous subsection showed how geodesics are morphed by the parameter $n(\mathbf{r})$. Here we note that without reference to $n(\mathbf{r})$, any coordinate morphing, i.e. $x(p) \to \bar{x}[\varphi(p)]$ transforms the connection coefficients to (see [16], p.36)

$$\bar{\Gamma}^{l}{}_{mn} = \left(\varphi_*\right)^{l}{}_{i}\left(\varphi^*\right)^{j}{}_{m}\left(\varphi^*\right)^{k}{}_{n}\Gamma^{i}{}_{jk} + \left(\varphi_*\right)^{l}{}_{j,k}\left(\varphi^*\right)^{j}{}_{m}\left(\varphi^*\right)^{k}{}_{n} \ , \tag{126}$$

where $\left(\varphi^*\right)^{i}{}_{l} = \partial\tilde{x}^i/\partial x^l$, and $\left(\varphi^*\right)^{i}{}_{l}\left(\varphi_*\right)^{l}{}_{j} = \delta^{i}{}_{j}$ (see Appendix B).

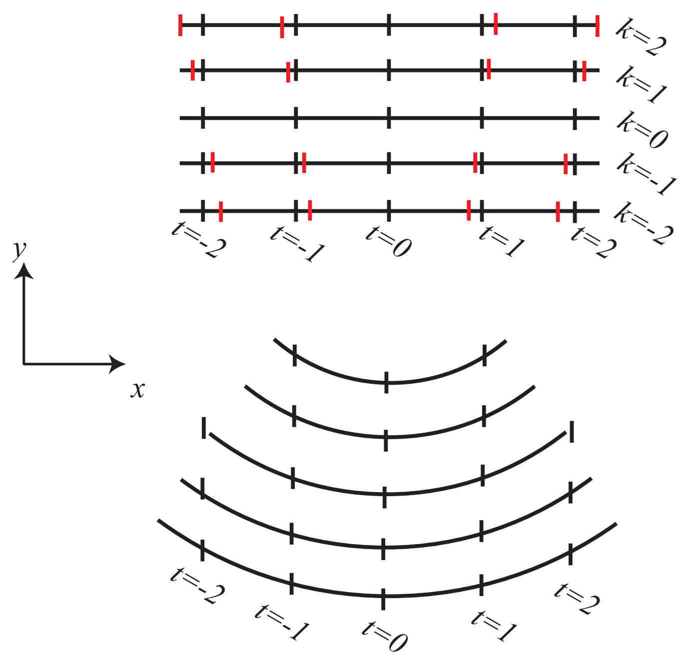

Figure 1: Illustration of the action of $-\frac{1}{2}\delta^{im}\delta_{jk}\left(\partial_m \ln n\right)$ (cf. Eq. (119)) in bending geodesics.

10.4 Combining Conformal-Transformation with Direct Transformation

The key result is now obtained by combining Eq. (115) with Eq. (126) to yield

$$\begin{aligned}\bar{\Gamma}^l{}_{mn} &= \left(\varphi_*\right)^l{}_i \left(\varphi^*\right)^j{}_m \left(\varphi^*\right)^k{}_n \Gamma^i{}_{jk} + \left(\varphi_*\right)^l{}_{j,k} \left(\varphi^*\right)^j{}_m \left(\varphi^*\right)^k{}_n \\ &= \frac{1}{2} g^{la} \left(\partial_m g_{an} + \partial_n g_{am} - \partial_a g_{mn}\right) \ ,\end{aligned} \tag{127}$$

where g_{ij} is determined from its Cartesian form given in Eq. (118). This links some desired deformation ($\varphi : p \to \varphi(p)$) of the geodesics of a homogeneous medium to the index distribution $n(\mathbf{r})$, of an inhomogeneous medium required to achieve that deformation.

10.5 Example: Acoustic Ray Cloak

As an application of Eq. (127) we consider again the 2-D cloaking transformation of Section Eq. (86) 8. In a polar coordinate basis, the non-zero connection coefficients for Euclidean space are given by

$$\Gamma^{r}{}_{\theta\theta} = -r \ , \quad \Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = r^{-1} \quad . \tag{128}$$

Also, from Eq. (86), we note

$$(\varphi^{*})^{r}{}_{r} = \frac{\partial r}{\partial \tilde{r}} = \frac{\tilde{r}}{(\tilde{r}^{2} - a^{2})^{1/2}} \quad , (\varphi_{*})^{r}{}_{r,r} = \frac{a^{2}}{\tilde{r}^{3}} \ . \tag{129}$$

The Cartesian metric $g_{ij} = n^2 \delta_{ij}$ becomes $g_{mn} = n^2 \mathrm{diag}(1, r^2)$ in polar coordinates, so that the non-zero connection coefficients are

$$\bar{\Gamma}^{r}{}_{rr} = \partial_r (\ln n) \ , \ \bar{\Gamma}^{r}{}_{\theta\theta} = -r \left[1 + r\partial_r (\ln n)\right] , \ \bar{\Gamma}^{\theta}{}_{r\theta} = r^{-1} + \partial_r (\ln n) \ . \tag{130}$$

Eq. (127) for $\Gamma^{r}{}_{rr}$ then turns out to be

$$\frac{d\,(\ln n)}{d\tilde{r}} = \frac{a^{2}}{\tilde{r}(\tilde{r}^{2} - a^{2})} \ , \tag{131}$$

which readily integrates up to

$$n = n_0 \left(1 - \frac{a^2}{\tilde{r}^2}\right)^{1/2} \quad , \tag{132}$$

where n_0 is the index of the initial homogeneous medium. Note that the index becomes zero at $r = a$ reflecting the fact that transport around the edge of the cloak must correspond to the null transit time across the pre-transformed origin. There is no causality violation as we are dealing with steady-state. Therefore to acoustically cloak an object immersed in a medium characterized by mass density ρ_0 and bulk modulus λ_0, the medium must be locally modified so that

$$\left(\frac{\rho}{\lambda}\right) = \left(\frac{\rho_0}{\lambda_0}\right) \left(1 - \frac{a^2}{r^2}\right) \ , \tag{133}$$

where the tilde has been dropped. The resultant cloaking function is illustrated in Fig. 2. The optical index cloak defined by Eq. (132) might

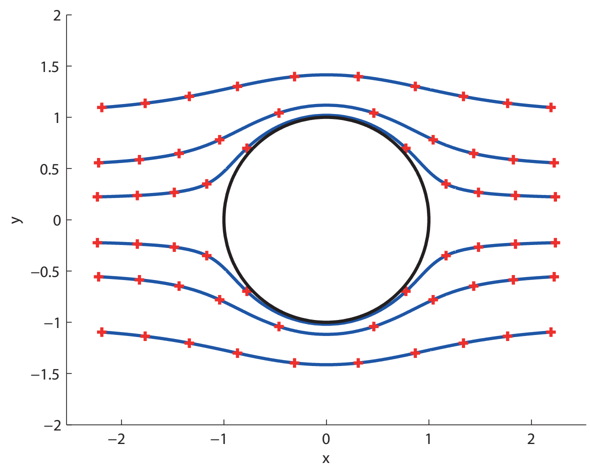

Figure 2: Acoustic ray cloaking function produced by the medium distribution described by Eq. (133). The red tick marks on each geodesic indicate time.

be achieved by surrounding the object to be cloaked with a semiconductor[4] with graded free-carrier density, $N(r)$. The refractive index is then approximately described by

$$n = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} , \tag{134}$$

where ω is the angular frequency of the incident light, and $\omega_p = \left(Ne^2/m\epsilon_0\right)^{1/2}$ is the plasma frequency determined from the free-carrier density, N. Equation (134) bears a striking similarity to Eq. (132). Provided always that $\omega > \omega_p$, the refractive index is real and less than unity. For an acoustic cloak, the initial homogeneous medium cannot be vacuum, and the required radial profile of the contrast in mass density and/or bulk modulus is determined from Eq. (133). Note also from Eq. (132) and Fig. 2 that the local phase velocity must become infinite at the cloak boundary $r = a$. Again this is unsurprising as the mapping of Eq. (86) takes the origin point, crossed in zero time by the original geodesics, to the points on the circle $r = a$.

[4]Note the structure must be band-gap engineered to prevent radial diffusion of the free carrier density.

11 Simple Covariant Acoustics

Here we present a simple but general description of scalar acoustics in a covarient-compatible form similar to tensor descriptions of electromagnetism. As noted previously (see Section 6) we use *first order* equations to model our wave mechanics [17], so that a pair of them are needed (in concert with constitutive or state equations) are needed to generate wave behaviour. If desired, the first order equations can be substituted inside one another to give a familiar second order form, but in fact the first order formulation is less restrictive. It is also more straightforward to apply a transformation scheme to the first order system.

A typical description of linearized acoustic waves [18] uses an an isotropic pressure field p, and fluid velocity field $\boldsymbol{v}$, linked by a bulk modulus λ and a density ρ. Here we extend the theretical description to explicitly allow for a momentum density field $\boldsymbol{V}$ and a dimensionless scalar field P; but for now choose to neglect the role of material velocity. In this way we can write down a simplified pseudo-acoustic (p-acoustic) form that is equivalent to a theory describing linear (i.e. perturbative) acoustic waves on a stationary background medium. The advantage for our purposes, however, is to emphasize both the similarities when compared to the electromagnetic description, which also uses two pairs of related quantities. Because we can use the same mathematical skeleton, transformation optics (T-optics) and transformation acoustics (T-acoustics) then are merged to become the same procedure. Only the construction of a particular transformation device (T-device) will differ between the two, being dependent on the expression of optics or acoustics within the overall machinery.

Moreover, p-acoustics is a more general model than the comparable traditional acoustic one, which typically reduces the equations back down to p and $\boldsymbol{v}$ – or even just a second order wave equation in p.

11.1 Vector Calculus Description

To make wave equations in the way we want, we need to link pairs of quantities by spatio-temporal differential equations, as well as by constitutive relations. Note that whilst a time derivative leaves a scalar as a scalar, and a vector as a vector, a spatial derivative can either promote a scalar to a vector, using the gradient, or demote a vector to a scalar, using the divergence[5].

[5] Another alternative is the curl operation, which takes a vector to a vector, as seen in EM.

Since the divergence $\nabla \cdot$ demotes the velocity field $\boldsymbol{v}$ to the change in a dimensionless scalar P, and the gradient operation ∇ promotes the scalar p to a temporal rate of change of vector momentum density $\boldsymbol{V}$, the p-acoustic system is

$$\partial_t P(\boldsymbol{r}, t) = \nabla \cdot \boldsymbol{v}(\boldsymbol{r}, t) + Q_P(\boldsymbol{r}, t), \tag{135}$$

$$\partial_t \boldsymbol{V}(\boldsymbol{r}, t) = \nabla p(\boldsymbol{r}, t) + \boldsymbol{Q}_V(\boldsymbol{r}, t), \tag{136}$$

where Q_P is a dimensionless source term for P, and $\boldsymbol{Q}_V$ a momentum density source term for $\boldsymbol{V}$.

These are not yet wave equations, however, since the two differential equations are independent. To link them, and create a wave theory, we define constitutive relations. In the most general case these are

$$P = \lambda^{-1} p - \boldsymbol{\alpha} \cdot \boldsymbol{v} \tag{137}$$

$$\boldsymbol{V} = \boldsymbol{\beta} p + \boldsymbol{\rho} \boldsymbol{v}, \tag{138}$$

where λ is a scalar bulk modulus relating the scalar p to the scalar P, and $\boldsymbol{\rho}$ is a matrix of densities relating the velocity field $\boldsymbol{v}$ to the momentum density $\boldsymbol{V}$. In addition there may also be unconventional couplings between the velocity field $\boldsymbol{v}$ and dimensionless P parameterized by a vector $\boldsymbol{\alpha}$, and another between the pressure field p and the momentum density parameterized by a vector $\boldsymbol{\beta}$.

11.2 Tensor Description

Whilst the vector calculus description above is accessible to a wide audience, we can gain considerable benefits by re-expressing it in a tensor form. This offers similar advantages to the re-expression of EM's Maxwell's equations in tensor form. For our purposes, the main advantage is that a transformation acoustics based on this prescription follows the same mathematical steps as for transformation optics. That is, one mathematical methodology can be used for both T-acoustics and T-optics, there is no need for separate and incompatible approaches.

So, just as for EM, we can now embed the field components $P, \boldsymbol{v}$ into a tensor $F^{\alpha\beta}$, and the $p, \boldsymbol{V}$ fields into a tensor density $G^{\alpha\beta}$; as well as populating a constitutive tensor χ with $\boldsymbol{\rho}, \lambda, \boldsymbol{\alpha}, \boldsymbol{\beta}$ appropriately. The p-acoustic wave mechanics will now look the same as EM, being

$$\partial_\alpha F^{\alpha\beta} = J^\beta, \qquad \partial_\beta G^{\alpha\beta} = K^\alpha . \tag{139}$$

with constitutive relations

$$G^{\alpha\beta} = \chi^{\alpha\beta}{}_{\gamma\delta} F^{\gamma\delta}, \tag{140}$$

and source terms J^α, K^α; but the internal structure – how the tensors are populated – will differ.

To see how the vector calculus quantities populate the tensor description, we utilise the Cartesian (t, x, y, z) coordinates. Then the tensor fields $F^{\alpha\beta}$ and $G^{\alpha\beta}$ can be written out as matrices

$$F^{\alpha\beta} = \begin{bmatrix} P & 0 & 0 & 0 \\ -v_x & 0 & 0 & 0 \\ -v_y & 0 & 0 & 0 \\ -v_z & 0 & 0 & 0 \end{bmatrix}, \quad G^{\alpha\beta} \quad = \begin{bmatrix} 0 & -V_x & -V_y & -V_z \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}, \tag{141}$$

together with the source terms $J^0 = Q_P$ and $\boldsymbol{K} = (K^i) = \boldsymbol{Q}_\nu$. In particular, note the repeated appearance of the single quantity p in three elements of $G^{\alpha\beta}$.

It is instructive to compare these p-acoustic field tensors with their EM counterparts, which are antisymmetric, have no repeated elements, and are only zero on the diagonal. In order to make the merger between acoustics and EM a seamless as possible, we use an unconventional choice of field tensor pair; i.e. the usual F but the dual of G:

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -cB_x & -cB_y & -cB_z \\ cB_x & 0 & E_z & -E_y \\ cB_y & -E_z & 0 & E_x \\ cB_z & E_y & -E_x & 0 \end{bmatrix}, \tag{142}$$

$$G^{\alpha\beta} = \begin{bmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & cH_z & -cH_y \\ -D_y & -cH_z & 0 & cH_x \\ -D_z & cH_y & -cH_x & 0 \end{bmatrix}. \tag{143}$$

One consequence of this, but not further discussed here, is that the matrix representation of the EM constitutive tensor becomes (block) off-diagonal.

11.3 Constitutive Relations

As with EM, the constitutive tensor has four indices, making a rewriting as a matrix problematic; so we follow the same approach as EM and compact it: we stack the non-zero elements of the $F^{\alpha\beta}$ to make a vector $F^{\mathcal{A}}$, and do likewise for $G^{\alpha\beta}$ to make a vector $G^{\mathcal{B}}$, However, because of the repeated element p in the $G^{\alpha\beta}$ tensor, the situation is less straighforward than for EM.

Explict form: The most explict Compaction contains redundant information, but will reveal post transformation inconsistencies most clearly. To do this we choose to compact the G tensor and preserve independently each of the diagonal elements G^{xx}, G^{yy}, and G^{zz} – even though each is simply the same quantity p. This helps us identify transformations or deformations of the coordinates that give rise to material properties incompatible with the vector model; i.e. where G^{xx}, G^{yy} and G^{zz} are not all identical. Thus the $F^{\mathcal{A}}$ and $G^{\mathcal{B}}$ column vectors are

$$\left[F^{\mathcal{A}}\right] = \left[P, v_x, v_y, v_z\right]^T , \tag{144}$$

$$\left[G^{\mathcal{B}}\right] = \left[p, p, p, V_x, V_y, V_z\right]^T . \tag{145}$$

The matrix representation of $\chi^{\alpha\beta}{}_{\gamma\delta}$ is not square, but still has the upper index denoting columns, and the lower index denoting rows. It has a matrix constitutive relation given by

$$\left[G^{\mathcal{A}}\right] = \left[\chi^{\mathcal{A}}{}_{B}\right]\left[F^{B}\right] , \tag{146}$$

$$\begin{bmatrix} G^{xx} \\ G^{yy} \\ G^{zz} \\ G^{xt} \\ G^{yt} \\ G^{zt} \end{bmatrix} = \begin{bmatrix} \lambda & \alpha_x & \alpha_y & \alpha_z \\ \lambda & \alpha_x & \alpha_y & \alpha_z \\ \lambda & \alpha_x & \alpha_y & \alpha_z \\ \beta_x & \rho_{xx} & \rho_{xy} & \rho_{xz} \\ \beta_x & \rho_{yx} & \rho_{yy} & \rho_{yz} \\ \beta_x & \rho_{zx} & \rho_{zy} & \rho_{zz} \end{bmatrix} \begin{bmatrix} F^{tt} \\ F^{tx} \\ F^{ty} \\ F^{tz} \end{bmatrix} . \tag{147}$$

At this point we might hope to generalize away from our original vector model and replace the scalar λ with a vector, and the vector $\boldsymbol{\alpha}$ into a matrix; but this would emphatically *not* be equivalent to the original p-acoustic equations in eqns. (135), (136). Further, if we were to try to design a T-device by transforming eqn. (147), whatever transforming we pick would have to preserve the repeated rows of $\left[\chi^{\mathcal{A}}{}_{B}\right]$ intact.

Abbreviated form: We can choose to use the fact that the elements G^{xx}, G^{yy} and G^{zz} are all just p, and so write a more compact vector for G using only (e.g.) G^{xx}. We now have a 4×4 constitutive matrix, with field column vectors G^{A}, F^{B},

$$\left[G^{A}\right] = \left[p, V_x, V_y, V_z\right]^T \tag{148}$$

$$\left[F^{B}\right] = \left[P, v_x, v_y, v_z\right]^T . \tag{149}$$

The matrix representation of $\chi^{\alpha\beta}{}_{\gamma\delta}$ has the upper index denoting columns, and the lower index denoting rows, so that the matrix constitutive relation is

$$\left[G^A\right] = \left[\chi^A{}_B\right]\left[F^B\right] \tag{150}$$

$$\begin{bmatrix} G^{cc} \\ G^{xt} \\ G^{yt} \\ G^{zt} \end{bmatrix} = \begin{bmatrix} \lambda & \alpha_x & \alpha_y & \alpha_z \\ \beta_x & \rho_{xx} & \rho_{xy} & \rho_{xz} \\ \beta_x & \rho_{yx} & \rho_{yy} & \rho_{yz} \\ \beta_x & \rho_{zx} & \rho_{zy} & \rho_{zz} \end{bmatrix} \begin{bmatrix} F^{tt} \\ F^{tx} \\ F^{ty} \\ F^{tz} \end{bmatrix}. \tag{151}$$

Note that we must remember that the G^{cc} element can be freely substituted by G^{xx}, G^{yy} or G^{zz}; this acts as an implicit constraint on allowed transformations, which are only allowed to transform x, y, and z equivalently. Thus any allowed non-isotropic wave behaviour has to derive from $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, or $\boldsymbol{\rho}$, and *not* from a transformation.

11.4 Covariance

This implementation of acoustic waves (p-acoustics) is written down in a covariant way; thus the mathematical description is intrinsically covariant – the form of the equations remains invariant under Lorentz transformations. However, note that although the mathematics is covariant, the underlying *physics* of acoustics is not, since the constitutive parameters λ, $\boldsymbol{\rho}$ are not frame independent, unlike the ϵ_0 and μ_0 in EM. Nevertheless, the covariant description mimics (as far as possible) the tensor representation of Maxwell's equations, which emphasises the similarity of the transformation mechanical machinery used to design T-devices in either T-acoustics or T-optics.

11.5 Pentamode Acoustics

Stress/shear acoustic waves can be modelled using the "pentamode material" model [9], based on 6-vectors for stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\epsilon}$, linked in a constitutive relation by a 6x6 matrix $\mathfrak{C}$ of elastic moduli. This theory reduces to an orientation vector $\boldsymbol{s}$ and two scalars that scale it to get stress and strain vectors. As a result, for small amplitude disturbances, the model can be reduced to a form suggestively similar to p-acoustics, but with the addition of an extra matrix $\mathfrak{S}$ encoding the anisotropic material parameters.

Thus, just as the p-acoustics vector calculus eqns. (135) and (136) can be inferred from traditional acoustics models, similar forms can be generated for pentamode acoustics. The relevant vector calculus re-representation of Norris's pentamode wave equations are

$$\partial_t \boldsymbol{V} = - \nabla \cdot \boldsymbol{\mathcal{S}} p \ , \tag{152}$$

$$\partial_t P = -\boldsymbol{\mathcal{S}} : \nabla \boldsymbol{v} \ , \tag{153}$$

with

$$\boldsymbol{V} = \boldsymbol{\rho v} + \boldsymbol{\beta} p \ , \tag{154}$$

$$P = \lambda^{-1} p + \boldsymbol{\alpha} \cdot \boldsymbol{v} \ , \tag{155}$$

where the pseudo-pressure p is derived from a negative single stress term as $p = -\lambda \mathrm{Tr}(\boldsymbol{\mathcal{S}}\epsilon)$. Note that this is a more general model than Norris's pentamode description, and is not an approximation of it.

In an indexed notation, these differential and constitutive equations are

$$\partial_t \boldsymbol{V}_i = - \nabla_k S^k{}_i p \ , \tag{156}$$

$$\partial_t P = -S^k{}_i \nabla^i \boldsymbol{v}_i \ , \tag{157}$$

with

$$V_i = \rho_i^j \boldsymbol{v}_j + \beta_i p \ , \tag{158}$$

$$P = \lambda^{-1} p + \boldsymbol{\alpha}^j \boldsymbol{v}_j . \tag{159}$$

Taking $\boldsymbol{\mathcal{S}}$ to be a constant, i.e. that these equations describe waves in a uniform medium. Since $\boldsymbol{\mathcal{S}}$ should have an inverse $\boldsymbol{\mathcal{S}}^{-1} = \boldsymbol{\mathcal{T}}$, we can pre-multiply the PM model's $\partial_t V_i$ equations,

$$T^i{}_l \partial_t V_i = -T^i{}_l \nabla_k S^k{}_i p \ , \tag{160}$$

$$\partial_t T^i{}_l V_i = - \nabla_k T^i{}_l S^k{}_i p \ , \tag{161}$$

$$\partial_t \bar{V}_l = - \nabla_k \delta^k{}_l p \ , \tag{162}$$

$$\partial_t \bar{V}_l = - \nabla_l p \ , \tag{163}$$

leaving us with a differential equation for $\bar{V}$ with identical structure to the basic p-acoustics $\partial_t \boldsymbol{V}$ equation; with

$$\bar{V}_l = T^i{}_l V_i \ , \qquad V_i = S_i{}^l \bar{v}_l \ . \tag{164}$$

Likewise the PM model's $\partial_t P$ equation can be adapted,

$$\partial_t P = -S^k{}_i \nabla^i v_k \tag{165}$$

$$\partial_t \pi = - \nabla^i S^k{}_i v_k \tag{166}$$

$$= - \nabla^i \bar{v}_i \ , \tag{167}$$

and again this leaves us with a differential equation for P with identical structure as the basic p-acoustics $\partial_t P$ equation, with

$$\bar{v}_i = S^k_{\ i} v_k; \qquad v_k = T_k^{\ i} \bar{v}_i \ . \tag{168}$$

So we see that the two vector wave properties $\boldsymbol{v}$ and $\boldsymbol{V}$ have been transformed. This affects the vector constitutive relation as follows:

$$V_i = \rho_i^{\ j} v_j + \beta_i p \ , \tag{169}$$
$$T^i_{\ l} V_i = T^i_{\ l} \rho_i^{\ j} T_k^{\ m} \bar{v}_m + T^i_{\ l} \beta_i p \ , \tag{170}$$
$$\bar{V}_i = \bar{\rho}_i^{\ m} \bar{v}_m + \bar{\beta}_i p \ , \tag{171}$$
$$\text{with} \qquad \bar{\rho}_i^{\ m} = T^i_{\ l} \rho_i^{\ j} T_k^{\ m} \ , \tag{172}$$
$$\text{and} \qquad \bar{\beta}_i = T_l^{\ i} \beta_l \ . \tag{173}$$

It also affects the scalar constitutive relation,

$$P = \lambda^{-1} p + \boldsymbol{\alpha}^i v_i \ , \tag{174}$$
$$= \lambda^{-1} p + \alpha^i T_i^{\ j} \bar{v}_j \ , \tag{175}$$
$$= \lambda^{-1} p + \bar{\alpha}^j \bar{v}_j \ , \tag{176}$$
$$\text{with} \qquad \bar{\alpha}^j = T^j_{\ i} \alpha^i \ . \tag{177}$$

Thus the transformed PM-acoustics equations are

$$\partial_t \bar{\boldsymbol{V}} = - \nabla p \ , \tag{178}$$
$$\partial_t P = - \nabla \cdot \bar{\boldsymbol{v}} \ , \tag{179}$$
$$\bar{\boldsymbol{V}} = \bar{\boldsymbol{\rho}} \bar{\boldsymbol{v}} + \bar{\boldsymbol{\beta}} p \ , \tag{180}$$
$$P = \lambda^{-1} p + \bar{\boldsymbol{\alpha}} \cdot \bar{\boldsymbol{v}} \ . \tag{181}$$

Thus the PM acoustic equations codify the same wave behaviour as the ordinary p-acoustic ones, albeit with the matrix $\boldsymbol{S}$ encoding a skew between the physical velocity field $\boldsymbol{v}$ and the effective one $\bar{\boldsymbol{v}}$; and betwwen the physical momentum density and the effective one $\bar{\boldsymbol{V}}$. Further, the density matrix $\boldsymbol{\rho}$, and cross-couplings $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ also have to be transformed to match.

11.6 Transformations

The transformations used in designing T-devices are usually expressed as coordinate transformations between an original set α with a known solution of the wave equations (usually a flat space solution), and a new set α' in which the solution is distorted to form the T-device, such as one with a central hole for use as a cloaking T-device. We can write down the Jacobian of the transformation between α and α as

$$T_{\alpha}^{\alpha'} = \left[\frac{\partial \alpha'}{\partial \alpha} \right], \tag{182}$$

so that the field density G and field F transform as

$$G^{\alpha'\beta'} = \{\det(T)\}^{-1} T_{\alpha}^{\alpha'} T_{\beta}^{\beta'} G^{\alpha\beta}, \qquad F^{\alpha'\beta'} \qquad = T_{\alpha}^{\alpha'} T_{\beta}^{\beta'} F^{\alpha\beta}, \tag{183}$$

and the rules of tensor calculus ensure that both the differential equation and constitutive relation works in any coordinate system:

$$\partial_{\alpha} F^{\alpha\beta} = J^{\beta}, \quad \partial_{\beta} G^{\alpha\beta} = K^{\alpha} \qquad \Leftrightarrow \qquad \partial_{\alpha'} F^{\alpha'\beta'} = J^{\beta'}, \quad \partial_{\beta'} G^{\alpha'\beta'} = K^{\alpha'} \tag{184}$$

$$G^{\alpha\beta} = \chi_{\gamma\delta}^{\alpha\beta} F^{\gamma\delta} \qquad \Leftrightarrow \qquad G^{\alpha'\beta'} = \chi_{\gamma'\delta'}^{\alpha'\beta'} F^{\gamma'\delta'}, \tag{185}$$

where

$$\chi_{\gamma'\delta'}^{\alpha'\beta'} = \frac{1}{\det(T)} T_{\alpha}^{\alpha'} T_{\beta}^{\beta'} \chi_{\gamma\delta}^{\alpha\beta} T_{\gamma'}^{\gamma} T_{\delta'}^{\delta}. \tag{186}$$

Of course there is an important caveat here: although the transformed constitutive relation and tensor $\chi_{\gamma'\delta'}^{\alpha'\beta'}$ are mathematically valid, the new $\chi_{\gamma'\delta'}^{\alpha'\beta'}$ might not have a form allowed by the physical model specified in the original theory – i.e. the physical χ is not a tensor, despite the mathematical notation. We will see how this happens in practise in the example that follows.

11.7 An example: the spatial carpet cloak

It is instructive to consider an example showing how a particular T-device might be implemented. Consider a carpet cloak, a means of concealing a bump in a planar reflective surface. Here this lies flat on the $y - z$ plane at $x = 0$, and reflects waves coming from the negative x half-space. We decide to transform a localized region of space reaching no higher than $x = -H$ above the plane and no further than $y = \pm\Lambda$ sideways. This is done to create a triangular cloaked region h high that is undetectable to any exterior observer, as shown on fig. 3.

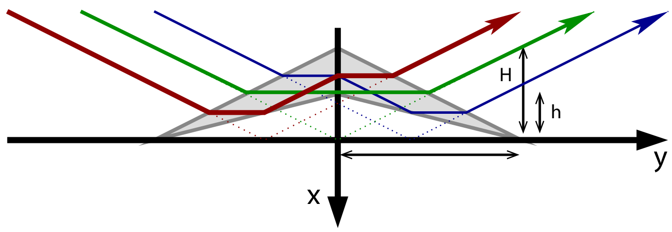

Figure 3: The (a) spatial carpet cloak. Continuous lines show the actual ray trajectories, dotted lines the apparent (illusiory) ray trajectories.

The deformation that achieves these T-devices is applied (only) inside the shaded regions on fig. 3, and is as follows:

$$t' = t, \qquad y' = y, \qquad z' = z, \tag{187}$$

$$x' = Rx - \frac{\Lambda - y.\,\mathrm{sgn}(y)}{\Lambda} h \qquad = Rx + h - rsy, \tag{188}$$

where $R = (H - h)/H$ is the ratio of actual depth to apparent depth, and $r = h/\Lambda$, and $s = \mathrm{sgn}(y)$.

The effect of this deformation from eqn. (188) is given by the tensor $L_\alpha^{\alpha'}$, which specifies the differential relationships between the primed and un-primed coordinates, where $\alpha \in \{t, x, y, z\}$ and $\alpha' \in \{t', x', y', z'\}$. If we let columns span the upper indices, and rows the lower, then in matrix form we have

$$\left[L_\alpha^{\alpha'}\right] = \left[\frac{\partial \alpha'}{\partial \alpha}\right] \qquad = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & -rs & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{189}$$

and note that $\det(L_\alpha^{\alpha'}) = R$.

For p-acoustic transformations, we need two compactified transformation matrices, with correctly arranged ordering. This is less straightforward than in EM, where rows and columns are indexed by the same list of coordinate pairs. To transform $G^{\alpha\beta}$, compactly indexed by cc, xt, yt, zt, we need $[N_A^{A'}]$

$$\left[G^{A'}\right] = \left[N_A^{A'}\right]\left[G^A\right] \tag{190}$$

$$\begin{bmatrix} G_{c'c'} \\ G_{x't'} \\ G_{y't'} \\ G_{z't'} \end{bmatrix} = \begin{bmatrix} \left\{R^2,\ 1\right\} & 0 & 0 & 0 \\ 0 & R & -rs & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} G_{cc} \\ G_{xt} \\ G_{yt} \\ G_{zt} \end{bmatrix}. \tag{191}$$

Note that the top left corner entry has *two* entries, i.e. R^2 if the cc index stands for xx, but 1 if yy or zz. This is the first explicit sign that all types of T-device are not possible with p-acoustics, and that in most cases restrictions will be imposed by the scalar nature of p-acoustics itself. For the moment we will continue with the ambiguity, so that we can determine how to resolve this at the final step when all consequences of the T-device design will be visible.

To transform $F^{\alpha\beta}$, compactly indexed by tt, tx, ty, tz, we need $[M_A^{A'}]$

$$\left[F^{A'}\right] = \left[M_A^{A'}\right]\left[F^A\right] \tag{192}$$

$$\begin{bmatrix} F_{t't'} \\ F_{t'x'} \\ F_{t'y'} \\ F_{t'z'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & R & -rs & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_{tt} \\ F_{tx} \\ F_{ty} \\ F_{tz} \end{bmatrix}, \tag{193}$$

The compact constitutive matrix transforms like,

$$[\chi_{B'}^{A'}] = \frac{[N_A^{A'}][\chi_B^A][M_B^{B'}]^T}{\det(L)}, \tag{194}$$

where a transpose operation has to be applied to the M matrix to ensure it represents the tensor summation correctly.

Hence

$$[\chi_{B'}^{A'}] = \frac{[N_A^{A'}][\chi_B^A][M_B^{B'}]^T}{\det(L)} = \begin{bmatrix} \{\lambda R,\ \lambda/R\} & 0 & 0 & 0 \\ 0 & \rho(R^2+r^2)/R & -rs\rho/R & 0 \\ 0 & -rs\rho/R & \rho/R & 0 \\ 0 & 0 & 0 & \rho/R \end{bmatrix}. \tag{195}$$

Just as for the N transforation matrix, we can see an ambiguity in constitutive matrix, i.e. in the modulus λ' which must be both λR and λ/R. This kind of problem is intrinsic to any T-device design for a scalar wave that scales different directions differently.

One point to note is that although the waves will propagate differently depending on our choice for λ', this is just a scalar multiple applied to the anisotropic properties specified by $\boldsymbol{\rho}'$, i.e. the 3x3 lower-right submatrix of χ' dependent on the reference density ρ. This means that although the ray trajectories will be correctly preserved in our (necessarily imperfect) carpet cloak, the wave speeds will not. Further, we cannot perfectly impedance match across the free-space to cloak interface, leading to reflections.

Only with non-trivial acoustic waves can we rescue this situation, at least to a degree; e.g. with pentamode waves we might restrict ourselves to cloaking in the 2D xy plane with the wave oscillations at right angles along z. We can achieve this means of a careful choice of the matrix $\mathbb{S}$ defining the stress-strain orientation, thus isolating the wave dynamics from the ambiguity in the constitutive matrix.

Alternatively, we could design a *corner-cloak*, the cloaking counter part to a corner cube reflector, but this is a rather trivial device, with each of x, y, z being scaled by the same amount with no cross-coupling, i.e. $c' = Rc + h$.

11.8 Discussion

The above description makes two important points. First, that acoustics and EM can use a unified description in which a single transformation theory can be defined without reference to either acoustics or optics specifically – although the implementation of any given T-device will be type dependent. To clarify this statement, for acoustics, which is not a priori pre-metric, there seem to be essentially three ways of defining a transformation scheme:

1. Instead of the exact theory, take an approximation, such as the Ray Acoustics of Section 10.

2. Despite the fact that the equations of the exact acoustic theory are not premetric, transform the metric dependent equations and try to find a transformation scheme at least for certain specific cases. This is what was done in Sections 6-8.

3. Since the equations of an exact acoustics theory are not premetric, we can aim to construct a new model which is premetric, as is done in this section. While this model is capable of being adapted directly for describing many types of acoustic wave in practical circumstances, it is important to note that it could only be mapped onto an exact metric-dependent acoustics in specific cases.

Secondly, the pentamode theory as finally applied by Norris as an example of an acoustic wave for which more general T-acoustic devices might be possible is not so – it is mathematically identical to ordinary p-acoustics. The similarity is guaranteed by the symmetry (and hence invertibility) of the stress-strain orientation matrix $\mathbb{S}$.

It is worth noting that there are other useful properties that follow from this p-acoustic wave theory. We can construct flux vectors analogous to the

Poynting vector of EM, which are simple products of one choice of scalar quantity (i.e. p or P) and one choice of vector quantity (i.e. $\boldsymbol{v}$ or $\boldsymbol{V}$). Just as for EM, there are four possibilities, each with their own continuity equation. Also, scalar wave potential(s) can be defined, with wave quantities being extracted by a time derivative or a gradient operation; as can vector wave potential(s), with wave quantities being extracted by a divergence operation or a time derivative.

As a final note, consider the linearized acoustics with anisotropic mass as discussed at the beginning of this report. The wave theory there is based on eqns. (19), (3), (5), i.e.

$$\begin{aligned} D_k \sigma_m{}^k &= g_{mn} \rho^n{}_l \ddot{u}^l, \\ e_{nk} &= \frac{1}{2} \left(D_k u_k + D_n u_k \right), \\ \sigma_i{}^j &= C_i{}^{jkl} e_{kl}. \end{aligned}$$

Now the usual linearized acoustics situation involves a bulk background medium with spatially and temporally constant parameters. This allows us to shift the time derivatives to define a material velocity field v_k and a momentum density V_m, and to redefine the density using the metric several times to raise and lower indices, to get

$$v_k = \partial_t u_k, \tag{196}$$

$$V_m = g_{mn} \rho^n{}_l \partial_t u^l = \bar{\rho}_{ml} v^l = \bar{\bar{\rho}}_m{}^k v_k. \tag{197}$$

These allow us to rewrite that wave theory in a manner similar to that used here for both p-acoustics, PM acoustics, and even EM; i.e. as pairs of quantities linked by constitutive relations, and cross-linked by differential equations. Taking the time derivative of the e_{nk} equation, inverting the $\sigma_i{}^j$ equation using a T such that $C_i{}^{jkl} T^i{}_{jkl} = 1$, we can achieve this. We find that the wave theory can be written using the differential equations

$$\begin{aligned} \partial_t V_m &= D_k \sigma_m{}^k, \\ \partial_t e_{nk} &= \frac{1}{2} \left(D_k + D_n \right) v_k, \end{aligned} \tag{198}$$

with the constitutive equations

$$V_m = \bar{\bar{\rho}}_m{}^k v_k \tag{199}$$

$$e_{kl} = T^i{}_{jkl} \sigma_i{}^j. \tag{200}$$

This theory interlinks a pair two-index quantities $\sigma_m{}^k, e_{nk}$ to a pair of single-index ones V_m, v_k; we might choose to express it as pairs of matrices and vectors. A new feature is the non-standard spatial derivative part in the $\partial_t e_{nk}$ equation, which here is a sum of two parts.

This demonstrates that the type of approach used in this section is valid even for treating acoustic wave theory of the kind discussed in the earlier part of this report.

A Anisotropic Mass Density: on the Correct Index Positioning of the Mass Tensor

Consider a 4-dimensional setting. Conservation of particle (baryon) number reads

$$dJ = 0 \ , \tag{201}$$

where J is the baryon current density 3-form, and it is a twisted quantity. Note that in Eq. (201), the symbol d represents the exterior derivative. Now impose the condition that

$$n \rfloor J = 0 \ , \tag{202}$$

where n is the tangent vector to the baryons' trajectories (i.e. n is the 4-velocity of the baryons). Note that in Eq. (A), the symbol $\rfloor$ represents the interior product.

$$n^\alpha J_{\alpha\beta\gamma} = 0 \ . \tag{203}$$

Note that in Eq. (203), the greek indices run over the four dimensions of spacetime. Now define

$$\check{J}^\alpha := \frac{1}{3!}\epsilon^{\alpha\beta\gamma\delta} J_{\beta\gamma\delta} \Rightarrow J_{\alpha\beta\gamma} = \hat{\epsilon}_{\alpha\beta\gamma\delta}\check{J}^\delta \ , \tag{204}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol. The quantity $\check{J}^\alpha$ is therefore a twisted vector density of weight $+1$. Then

$$n^\alpha \hat{\epsilon}_{\alpha\beta\gamma\delta}\check{J}^\delta = 0 \quad \Leftrightarrow \quad \hat{\epsilon}_{\alpha\beta\gamma\delta} n^\gamma \check{J}^\delta = 0 \ . \tag{205}$$

Contract with $\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}$ as follows:

$$\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\hat{\epsilon}_{\alpha\beta\gamma\delta} n^\gamma \check{J}^\delta = 0 \ . \tag{206}$$

Recall that $\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\hat{\epsilon}_{\alpha\beta\gamma\delta} = \delta^{\mu\nu}_{\gamma\delta}$, and obtain

$$\delta^{\mu\nu}_{\gamma\delta} n^\gamma \check{J}^\delta = 0 \quad \Leftrightarrow \quad n^{[\mu}\check{J}^{\nu]} = 0 \quad \Leftrightarrow \quad \check{J}^\nu = \eta n^\nu \ , \tag{207}$$

where η is a twisted scalar density of weight $+1$, and represents the density of baryons (see [12]). In 3-dimensions, the density of mass is a twisted 3-form. Thus, in 4-dimensions, there must exist a current density 3-form K, which is twisted. Hence the mass tensor is a linear map relating K and J. More precisely, in the non-relativistic limit in which mass does not depend on speed,

$$K = m(J) \ . \tag{208}$$

Since J is dimensionless, and K has units of mass, then $[m] = [\text{mass}]$. Also, m is untwisted. In tensor-index notation

$$K_{\alpha\beta\gamma} = \frac{1}{3!} m_{\alpha\beta\gamma}{}^{\mu\nu\lambda} J_{\mu\nu\lambda} \ , \tag{209}$$

where $m_{\alpha\beta\gamma}{}^{\mu\nu\lambda}$ is a true tensor. Note that:

$$m_{[\alpha\beta\gamma]}{}^{\mu\nu\lambda} = m_{\alpha\beta\gamma}{}^{\mu\nu\lambda} \ , \tag{210}$$

and

$$m_{\alpha\beta\gamma}{}^{[\mu\nu\lambda]} = m_{\alpha\beta\gamma}{}^{\mu\nu\lambda} \ . \tag{211}$$

Define:

$$\check{K}^{\alpha} := \frac{1}{3!} \epsilon^{\alpha\beta\gamma\delta} K_{\beta\gamma\delta} \quad \Rightarrow \quad K_{\alpha\beta\gamma} = \hat{\epsilon}_{\alpha\beta\gamma\delta} \check{K}^{\delta} \ . \tag{212}$$

Evidently,

$$\hat{\epsilon}_{\alpha\beta\gamma\delta} \check{K}^{\delta} = \frac{1}{3!} m_{\alpha\beta\gamma}{}^{\mu\nu\lambda} \hat{\epsilon}_{\mu\nu\lambda\tau} \check{J}^{\tau} \ . \tag{213}$$

Contract through by $\frac{1}{3!}\epsilon^{\ \alpha\beta\gamma}$:

$$\frac{1}{3!} \epsilon^{\theta\alpha\beta\gamma} \hat{\epsilon}_{\alpha\beta\gamma\delta} \check{K}^{\delta} = \left(\frac{1}{3!}\right)^2 \epsilon^{\theta\alpha\beta\gamma} m_{\alpha\beta\gamma}{}^{\mu\nu\lambda} \hat{\epsilon}_{\mu\nu\lambda\tau} \check{J}^{\tau} \ , \tag{214}$$

so that

$$\check{K}^{\theta} = \check{m}^{\theta}{}_{\tau} \check{J}^{\tau} \ , \tag{215}$$

where $\check{m}^{\theta}{}_{\tau}$ is a true $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ tensor, defined via

$$\hat{m}^{\mu}{}_{\nu} := \left(\frac{1}{3!}\right)^2 \epsilon^{\mu\alpha\beta\gamma} m_{\alpha\beta\gamma}{}^{\eta\theta\tau} \hat{\epsilon}_{\eta\theta\tau\nu} \ . \tag{216}$$

Also

$$\check{K}^{\mu} = \check{m}^{\mu}{}_{\nu} \check{J}^{\nu} = \eta \check{m}^{\mu}{}_{\nu} n^{\nu} \ , \tag{217}$$

or

$$\begin{bmatrix} \eta \check{m}^{0}{}_{0} & 0 \\ 0 & \eta \check{m}^{i}{}_{j} \end{bmatrix} = \eta \check{m}^{\mu}{}_{\nu} \ . \tag{218}$$

The term on the bottom right of this equation is what Norris [9] calls $\sqrt{g}\rho^{i}{}_{j}$.

B Push-forward (φ_*) and Pull-back (φ^*) Notation,

Central to any transformation scheme is the mapping of one manifold M, to another manifold N, i.e. $\varphi : M \to N$. For various technical reasons, we assume that the mapping is a *diffeomorphism*, so that φ is invertible, and $M \sim N$ (i.e. M is diffeomorphic to N). Let x^i represent coordinates in M and $\bar{x}^j$ coordinates in N. For a given vector field u in M, φ induces a vector field $\bar{u} = \varphi_* u$ in N (the *push-forward* of u) via the standard rules for coordinate transformation:

$$(\varphi_* u)^j = (\varphi_*)^j{}_i u^i \ , \tag{219}$$

where $(\varphi_*)^j{}_i = \partial \bar{x}^j / \partial x^i$, and the summation convention has been used. Likewise, since φ is a diffeomorphism, the *pull-back* $\varphi^* w$ of a vector $w \in N$ may be defined via

$$(\varphi^* w)^i = (\varphi^*)^i{}_j w^j \ , \tag{220}$$

where $(\varphi^*)^i{}_j = \partial x^i / \partial \bar{x}^j$, and $(\varphi^*)^i{}_k (\varphi_*)^k{}_j = \delta^i{}_j$. In this notation $\bar{g}^{mn} = (\varphi_* g)^{mn}$ are the pushed forward components of a $(2,0)$ tensor whose components in M are g^{ij}. They are given by

$$(\varphi_* g)^{mn} = (\varphi_*)^m_i \, (\varphi_*)^n_j \, g^{ij} \ . \tag{221}$$

For a mixed second rank tensor[6] in M whose components are given by $\rho^i{}_j$, the corresponding pushed forward components in N are $\bar{\rho}^m_{\ n} = (\varphi_* \rho)^m_{\ n}$, and are given by

$$(\varphi_* \rho)^m_{\ n} = (\varphi_*)^m_i \, (\varphi^*)^j{}_n \, \rho^i{}_j \ . \tag{222}$$

Finally[7], a *function* ξ on M may be pushed forward to $\bar{\xi} = \varphi_* \xi$ on N via

$$\varphi_* \xi \left[\varphi(p) \right] = \xi(p) \ , \tag{223}$$

where $p \in M$.

C Proof of Inertial Cloaking Theorem

This appendix gives the proof of the theorem stated in Section 6.

[6] For a mixed tensor the distinction between push-forward and pull-pack is blurred, and so the following definition should be considered a formal one which is only defined when φ is a diffeomorphism.

[7] Again, this definition relies on the invertibility of φ.

Part 1:

$$\lambda'\xi' = \frac{\sqrt{\det \bar{g}_{ij}}}{\sqrt{\det g'_{ij}}}\bar{\xi}\frac{\sqrt{\det g'_{ij}}}{\sqrt{\det \bar{g}_{ij}}} = \bar{\xi}\bar{\lambda} \ .$$

Part 2:

$$g'_{ij}\rho'^{j}{}_{k}\ddot{u}'^{k} = g'_{ij}\frac{\sqrt{\det g'_{ij}}}{\sqrt{\det \bar{g}_{ij}}}g'^{jm}\bar{g}_{mn}\bar{\rho}^{n}{}_{k}\ddot{u}'^{k} = \frac{\sqrt{\det g'_{ij}}}{\sqrt{\det \bar{g}_{ij}}}\bar{g}^{i}{}_{n}\bar{\rho}^{n}{}_{k}\frac{\sqrt{\det \bar{g}_{ij}}}{\sqrt{\det g'_{ij}}}\ddot{\bar{u}}^{k} = \bar{g}_{ij}\bar{\rho}^{j}{}_{k}\ddot{\bar{u}}^{k} \ .$$

Hence

$$g'_{ij}\rho'^{j}{}_{k}\ddot{u}'^{k} = \bar{g}_{ij}\bar{\rho}^{j}{}_{k}\ddot{\bar{u}}^{k} \ .$$

Part 3: On account of parts 1 and 2

$$\partial_i\left(\lambda'\xi'\right) = g'_{ij}\rho'^{j}{}_{k}\ddot{u}'^{k} \quad \Rightarrow \quad \partial_i\left(\bar{\lambda}\bar{\xi}\right) = \bar{g}_{ij}\bar{\rho}^{j}{}_{k}\ddot{\bar{u}}^{k} \ .$$

Part 4:

$$\xi' = \frac{1}{\sqrt{\det g'_{ij}}}\frac{\partial}{\partial x^p}\left(\sqrt{\det g'_{ij}}u'^p\right)$$

Hence

$$\frac{\sqrt{\det \bar{g}_{ij}}}{\sqrt{\det g'_{ij}}}\bar{\xi} = \frac{1}{\sqrt{\det g'_{ij}}}\frac{\partial}{\partial x^p}\left(\sqrt{\det g'_{ij}}\frac{\sqrt{\det \bar{g}_{ij}}}{\sqrt{\det g'_{ij}}}\bar{u}^p\right) \ ,$$

$$\bar{\xi} = \frac{1}{\sqrt{\det \bar{g}_{ij}}}\frac{\partial}{\partial x^p}\left(\sqrt{\det \bar{g}_{ij}}\bar{u}^p\right) \ .$$

Part 5:

$$\partial_i\left(\lambda'\xi'\right) = g'_{ij}\rho'^{j}{}_{k}\ddot{u}'^{k} \quad \Rightarrow \quad \partial_i\left(\bar{\lambda}\bar{\xi}\right) = \bar{g}_{ij}\bar{\rho}^{j}{}_{k}\ddot{\bar{u}}^{k} \quad \Rightarrow \quad \partial_i\left(\lambda\xi\right) = g_{ij}\rho^{j}{}_{k}\ddot{u}^{k} \ ,$$

and

$$\xi' = \frac{1}{\sqrt{\det g'_{ij}}}\frac{\partial}{\partial x^p}\left(\sqrt{\det g'_{ij}}u'^p\right) \quad \Rightarrow \quad \bar{\xi} = \frac{1}{\sqrt{\det \bar{g}_{ij}}}\frac{\partial}{\partial x^p}\left(\sqrt{\det \bar{g}_{ij}}\bar{u}^p\right)$$

$$\Rightarrow \quad \xi = \frac{1}{\sqrt{\det g_{ij}}}\frac{\partial}{\partial x^p}\left(\sqrt{\det g_{ij}}u^p\right) \ , \quad \text{QED.}$$

D Proof of Theorem 2 (Section 7.1)

Here we give the proof of the second theorem given in Section 7.1 (cf. Eqs. (81)-(85)). As previously, we divide the proof into several steps.

Step 1

$$\lambda' \xi' = \frac{\sqrt{\det g'_{pq}}}{\sqrt{\det \bar{g}_{pq}}} \bar{\lambda} \ \frac{\sqrt{\det \bar{g}_{pq}}}{\sqrt{\det g'_{pq}}} \bar{\xi} = \bar{\lambda} \bar{\xi} \ . \tag{224}$$

Step 2

$$S'_{ij} \rho'^{j}{}_{k} g'^{kl} S'_{lm} \ddot{w}'^{m} = \frac{\sqrt{\det g'_{pq}}}{\sqrt{\det \bar{g}_{pq}}} \bar{S}_{ia} \bar{\rho}^{a}{}_{b} \bar{g}^{bc} \bar{S}_{cm} \frac{\sqrt{\det \bar{g}_{pq}}}{\sqrt{\det g'_{pq}}} \ddot{\bar{w}}^{m} \ , \tag{225}$$

Step 3

Combining Step 1 and Step 2 implies that if

$$\partial_i \left(\lambda' \xi' \right) S'_{ij} \rho'^{j}{}_{k} g'^{kl} S'_{lm} \ddot{w}'^{m} \ , \tag{226}$$

then

$$\partial_i \left(\bar{\lambda} \bar{\xi} \right) \bar{S}_{ij} \bar{\rho}^{j}{}_{k} \bar{g}^{kl} \bar{S}_{lm} \ddot{\bar{w}}^{m} \ . \tag{227}$$

Step 4

If

$$\xi' = \frac{1}{\sqrt{\det g'_{pq}}} \partial_i \left(\sqrt{\det g'_{pq}} w'^{i} \right) \ , \tag{228}$$

then

$$\frac{\sqrt{\det \bar{g}_{pq}}}{\sqrt{\det g'_{pq}}} \bar{\xi} = \frac{1}{\sqrt{\det g'_{pq}}} \partial_i \left(\sqrt{\det g'_{pq}} \frac{\sqrt{\det \bar{g}_{pq}}}{\sqrt{\det g'_{pq}}} \ \ \bar{w}^{i} \right) \ . \tag{229}$$

Hence

$$\bar{\xi} = \frac{1}{\sqrt{\det \bar{g}_{pq}}} \partial_i \left(\sqrt{\det \bar{g}_{pq}} \bar{w}^{i} \right) \ . \tag{230}$$

Step 5

Combining the results of Step 3 and Step 4 yields

$$\partial_i\left(\lambda'\xi'\right) = S'_{ij}\rho'^{j}{}_{k}g'^{kl}S'_{lm}\ddot{w}'^{m}\ , \tag{231}$$

$$\Leftrightarrow \partial_i\left(\bar{\lambda}\bar{\xi}\right) = \bar{S}_{ij}\bar{\rho}^{j}{}_{k}\bar{g}^{kl}\bar{S}_{lm}\ddot{\bar{w}}^{m}\ , \tag{232}$$

$$\Leftrightarrow \partial_i\left(\lambda\xi\right) = S_{ij}\rho^{j}{}_{k}g^{kl}S_{lm}\ddot{w}^{m}\ , \tag{233}$$

and

$$\xi' = \frac{1}{\sqrt{\det g'_{pq}}}\partial_i\left(\sqrt{\det g'_{pq}}w'^{i}\right)\ , \tag{234}$$

$$\Leftrightarrow \bar{\xi} = \frac{1}{\sqrt{\det \bar{g}_{pq}}}\partial_i\left(\sqrt{\det \bar{g}_{pq}}\bar{w}^{i}\right)\ , \tag{235}$$

$$\Leftrightarrow \xi = \frac{1}{\sqrt{\det g_{pq}}}\partial_i\left(\sqrt{\det g_{pq}}w^{i}\right)\ . \tag{236}$$

QED.

References

[1] J.B. Pendry, D. Schurig and D.R.Smith, "Controlling electromagnetic fields", *Science*, **312**, 1780-1782, (2006).

[2] U. Leonhardt, "Optical conformal mapping", *Science*, **312**, 1777-1780, (2006).

[3] D. Schurig, J.J. Mock, B. J. Justice, S. A. Cummer, J. B. Pendry, A. F. Starr and D. R. Smith, *Science*, **314**, 977-980, (2006).

[4] R. T. Thompson, S. A. Cummer and J. Frauendiener, "A completely covariant approach to transformation optics", *J. Opt.*, **13**, 024008, (2011).

[5] R. T. Thompson, S. A. Cummer and J. Frauendiener, *J. Opt.*, "Generalized transformation optics of linear materials", **13**, 055105, (2011).

[6] G. W. Milton GW, M. Briane and J. R. Willis, "On cloaking for elasticity and physical equations with a transformation invariant form", *New J. Phys.* **8**, 248, (2006).

[7] S. A. Cummer and D. Schurig. "One path to acoustic cloaking", *New Journal of Physics*, **9**, 45, (2007).

[8] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, "Full-Wave Invisibility of Active Devices at All Frequencies", it Commun. Math. Phys., **275**, 749789, (2007).

[9] A. N. Norris, "Acoustic cloaking theory", *Proc. Roy. Soc. A*, **464**, 2411-2434, (2008).

[10] A. N. Norris, "Acoustic metafluids", *J. Acoust. Soc. Am.*, **125**, 839-849, (2009).

[11] F. Hehl and Y. Obukhov, *Foundations of Classical Electrodynamics*, vol. 33 of Progress in Mathematical Physics. Birkh´auser, Boston, 2003.

[12] A. Yavari, *Journal of Mathematical Physics*, **49**, 022901, (2008).

[13] Amenzade, Y., *Theory of Elasticity*, MIR publishers, (1979).

[14] L.D. Landau and E.M. Lifschitz, *The classical theory of fields, Volume 2*, Butterworth-Heinemann, (1975).

[15] Milton, G.W. and Cherkaev, A.V., "Which elasticity tensors are realizable?", *Journal of Engineering Materials and Technology*, 117, 483–493, (1995).

[16] E. J. Post, *Formal Structure of Electromagnetics*, Dover, (1997).

[17] R. Geroch, *Partial Differential Equations of Physics*, Proceedings of the Forty-Sixth Scottish Summer School in Physics, eds. G. S. Hall; J. R. Pulham, 1996.

[18] P. M. Morse, K. U. Ingard, *Theoretical Acoustics*, McGraw-Hill, New York, 1968.