Time transformations, anisotropy and analogue transformation elasticity

Final Report

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1 Introduction

Metamaterials offer an unprecedented flexibility in the construction of media with properties that are difficult or impossible to find in nature [1]. This concept first appeared within the frame of electromagnetism, and enabled scientists to design exotic devices such as negative-index superlenses [2]. Afterwards, the notion of metamaterial has been extended to other branches of physics [3], such as acoustics [4, 5], electronics [6] or thermodynamics [7,8]. To take full advantage of this flexibility in the synthesis of tailor-made properties, we also need new design techniques that help us to engineer these properties with the aim of building novel devices with advanced functionalities. Along this line, one of the most powerful techniques is transformation optics, which prescribes the properties that a medium should have in order to alter the propagation of light in almost any imaginable way [9–12]. As a result, metamaterials and transformation optics have teamed up to open the door to the realization of photonic devices that were unthinkable only a few years ago, such as invisibility cloaks or optical wormholes [13, 14], constituting one of the most interesting recent developments in material science. The great success of the transformational paradigm in the field of electromagnetism has led the research community to look for ways in which this approach could be extended to other fields [7, 15, 16].

Noticing that the key to transformation optics is the form invariance of Maxwell's equations under any spacetime transformation, the initial approach was to try to exploit form invariance in the governing equations of different physical phenomena. Therefore, one of the crucial issues in transformational methods is the range of coordinate transformations over which the relevant field equations have this property [17–20]. Outside of optics, acoustics is probably the field in which the greatest advance has been achieved. There, the form invariance of the acoustic equations under spatial transformations has been used to obtain the material parameters that deform acoustic space in the desired way, e.g., for cloaking acoustic waves [4, 16, 21–27].

However, this approach to transformation acoustics has been undermined by the deep structural differences between Maxwell's theory with its underlying relativistic geometry on the one hand, and the Galilean character of fluid mechanics on the other hand, which reduces the power of the traditional transformational method when applied to acoustics, which we will refer to as Standard Transformation Acoustics (STA). Specifically, classical acoustic equations are not form invariant under transformations that mix space and time [20]. As a consequence, the method cannot be applied to design devices based on this kind of transformation, contrarily to what has been done in optics [28–30].

Recently, the problem of transformation acoustics was approached from another angle [20, 31]. Instead of using directly the symmetries of the acoustic equations to bridge between different solutions for the propagation of acoustic waves, the symmetries of an analogue abstract spacetime (described by relativistic form-invariant equations) are exploited. In this method, each couple of solutions connected by a general coordinate transformation in the analogue spacetime can be mapped to acoustic space. This way, it is possible to find the relation between the acoustic material parameters associated with each of these transformation-connected solutions. The result is an alternative version of transformation acoustics as powerful as its optical counterpart and that we refer to as analogue transformations acoustics (ATA).

However, some important issues still remain to be clarified. In particular, we would like to explicitly obtain the set of transformations under which the acoustic equations are directly form-invariant in the original acoustic laboratory spacetime, since, in all the transformations that fall outside this set, the construction of the auxiliary relativistic analogue spacetime, and hence the use of ATA, is essential to achieve the desired transformation. Second, given that ATA and STA start from different initial equations (STA relies on pressure equations, whereas ATA starts from the velocity potential), another question that arises is whether it could be possible to construct an analogue transformation method based on the pressure wave equation, rather than the velocity potential formulation, and what its range of application would be. Third, being a very recent result, the potential applications of ATA for the construction of spacetime-mixing devices remain unexplored. For instance, it would be desirable to study if it is possible to design, through the use of the ATA technique, the acoustic counterparts of some spacetime electromagnetic transformation media, such as a frequency shifter, a spacetime compressor, or a spacetime cloak. Finally, since the medium properties appearing in the standard velocity potential equation are isotropic, the first version of ATA only admitted the possibility of working with isotropic transformations and a generalization to anisotropic ones is lacking. The first goal of this study is to address all these ATA-related issues.

Apart from acoustics, a second important field where transformation techniques have been hampered by the structure of the relevant field equations is in elasticity. The standard spatial equations of linear elasticity can be written tensorially (and hence form-invariant). However, this does not imply that this form-invariance is still valid for composite materials, or for transformations that require space-time mixing. In fact, based on Willis' work (see [32] and references therein), it is usually accepted that the traditional elastic wave equations are not form-invariant under general coordinate transformations (space-time mixing, but also purely spatial) when the elastic material is composite, let alone random [17, 18, 33]. Given the success of the analogue transformation method in acoustics just described, it is then natural to ask whether similar analogue transformation techniques could be relevant for elasticity as well. For the case of elasticity, our study is necessarily much more preliminary than for acoustics: The field of transformation elasticity is much less developed than the one of Transformation Acoustics, and there is no "off-the-shelf" analogue relativistic model available for elasticity. Our study of transformation elasticity therefore focuses on fundamental issues. First, we will analyze the traditional elastic equations and its behaviour under both spatial and space-time transformations. Then, we will discuss several possible generalizations (relativistic and other) of traditional elasticity that could pave the way for a "Transformation Elasticity" framework with a wide range of applications, and identify the key issues and open problems in each of these possible generalizations.

The results that have been obtained have application in a variety of technological sectors. The most obvious application to the space industry is in the payload/launcher vibration control. It is well known that acoustic and elastic vibrations during the launch phase can contribute to up to forty percent of the failure rate of the onboard systems. So far the main technological response to these problems has been the use of different kind of passive absorbers base on plastic materials which are lightweight and have an high dissipation rate. The techniques that we have developed allow in principle the construction of active devices which are able to control the vibrational energy, reducing its impact on the payload. In addition, metamaterial based systems have the advantage to be usable multiple times because the control of the vibrations is not obtained at the expense of the structure of the control systems.

The report is divided into two main blocks. In the first one (section 2), we describe in detail our work on analogue transformation acoustics. Specifically, in section 2.1 we derive the set of transformations under which the acoustic equations are form-invariant (section 2.1.1) and analyze the possibility of constructing an ATA technique based on the pressure wave equation (section 2.1.3). For completeness, we also provide a review on ATA in section 2.1.2. In section 2.2, we propose and design three new devices based on ATA, namely, a frequency shifter, a spacetime compressor and a spacetime cloak. Finally, in section 2.3 we extend the ATA method to anisotropic transformations and design a device able to cloak the acoustic velocity potential.

The second block (section 3) is devoted to our work on transformation elasticity. In section 3.1, we study the standard elastic wave equation and its spatial transformation properties. We extend this to spacetime transformations in section 3.2, where we suggest an approach through an abstract relativistic system. Section 3.3 is devoted to the Willis equations and their potential role in transformation elasticity, as proposed by Milton et al. [17]. In section 3.4 we discuss the possibility of further generalizing Milton et al.'s proposal to implement spacetime transformations. This leads to a natural interpretation of the Norris-Shuvalov setup for transformation elasticity [18] in terms of an Analogue Transformation-like framework.

Finally, we draw some conclusions in section 4.

2 Analogue Transformation Acoustics: anisotropic and spacetime transformations

In this block we report on our results concerning the first part of the project. We start by analyzing the properties of the acoustic equations under spacetime transformations. Then, we describe the design of two novel devices based on ATA, whose functionality is verified through numerical simulations. Finally, we extend the ATA method to the anisotropic case with the help of a homogenization technique.

2.1 Spacetime transformations of the acoustic equations

Our study begins by determining the set of transformations under which the acoustic wave equation is form invariant. This is an important piece of knowledge, as it clearly differentiates the cases in which both STA and ATA can be used from the ones in which only the second is valid. Afterwards, we provide a brief review of the ATA method, which will be used in the subsequent sections. We conclude this section by analyzing the possibility of constructing an ATA technique based on transformations of the pressure, which are physically different from those of the velocity potential and could result in alternative transformational acoustic media.

2.1.1 Form-invariance in the acoustic equations

The various existing analyses in STA start from the following basic equation for the pressure perturbations p_1 of a (possibly anisotropic) fluid medium: [34]

$$\ddot{p}_1 = B\nabla_i \left(\rho^{ij}\nabla_j p_1\right) \ . \tag{1}$$

Here, B is the bulk modulus and ρ^{ij} the (in general, anisotropic) inverse matrix density of the background fluid. We will use latin spatial indices (i, j) and Greek spacetime indices (μ, ν, with $x^0 = t$). This is a Newtonian physics equation so that ∇ represents the covariant derivative of the Newtonian flat 3-dimensional space. In generic spatial coordinates it will read

$$\ddot{p}_1 = B\frac{1}{\sqrt{\gamma}}\partial_i \left(\sqrt{\gamma}\rho^{ij}\partial_j p_1\right) \ , \tag{2}$$

where γ is the determinant of the three-dimensional spatial metric γ_{ij} (with γ^{ij} its inverse). The success of STA relies on the form invariance of this equation under spatial coordinate transformations. It is easy to prove, however, that Eq. (2) is not form invariant for more general (space-time mixing) transformations.

Another commonly used equation in acoustics is the one describing the evolution of the potential function ϕ_1 for the velocity perturbation $\mathbf{v}_1$ defined as $\mathbf{v}_1 = -\nabla\phi_1$ [35,36][1]:

$$-\partial_t \left(\rho c^{-2} \left(\partial_t\phi_1 + \mathbf{v}\cdot\nabla\phi_1\right)\right) + \nabla\cdot\left(\rho\nabla\phi_1 - \rho c^{-2}\left(\partial_t\phi_1 + \mathbf{v}\cdot\nabla\phi_1\right)\mathbf{v}\right) = 0, \tag{3}$$

where $\mathbf{v}$ is the background velocity, ρ the isotropic mass density and c the local speed of sound ($B = \rho c^2$). This equation is in many cases equivalent to Eq. (2), but it is constructed using other, at first sight less stringent, assumptions (see also section 2.3.2) and naturally includes the velocity $\mathbf{v}$ of the background fluid. Therefore one could construct a transformation acoustics method based on this equation, which contains this additional degree of freedom $\mathbf{v}$. In spite of this interesting feature, the use of Eq. (3) does not solve the problem of obtaining a transformation approach able to operate with spacetime transformations, since this equation is not invariant under general spacetime transformations either. Due to its complexity, it is not straightforward to see the exact set of transformations that do or do not preserve the form of Eq. (3). The first contribution of this work is the explicit derivation of these sets. By applying a

[1]Note that this definition does not impose any restriction on the vorticity of the background flow. In fact, even when the fluid is rotational, the present formalism can be maintained for sound waves satisfying $\omega \gg \omega_0$, with ω_0 the rotation frequency of the background fluid and ω that of the acoustic perturbation. See the discussion in [37].

generic spacetime coordinate transformation to Eq. (3), we can determine under which circumstances the form of this equation is preserved, without the appearance of new terms (which it would not be possible to interpret in terms of material parameters within the standard procedure). Let us start by taking a specific acoustic equation for the velocity potential. For convenience let us write it in the form

$$\sqrt{-f}\partial_\mu f^{\mu\nu}\partial_\nu\phi = 0 \ . \tag{4}$$

The coefficients $f^{\mu\nu}(t,\mathbf{x})$ have a specific functional dependence in the Cartesian coordinates (t,x). We have arranged them in the form of the inverse of a matrix array $f_{\mu\nu}$ and f represents the determinant of this matrix of coefficients.

The equation does not incorporate by itself properties associated to changes of coordinates. This is true because without additional information we don't know the transformation properties of the coefficients. For instance, the $f^{\mu\nu}(t,\mathbf{x})$ could be just an array of scalars, then a transformation of coordinates will involve only to take due care of the derivatives. The equation of acoustics we are dealing with comes from an initially Newtonian system. That is why we know that f^{00} is a scalar, f^{0i} a vector and f^{ij} a tensor, all under spatial coordinate transformations. Time is an external independent parameter. Under changes of the time parameter all the coefficients should transform as scalars. Recall also that the field ϕ is a scalar under any change of coordinates.

Let us perform a general transformation of the acoustic equation to see its new form. Consider the form

$$\partial_t f^{00}\partial_t\phi + \partial_t f^{0i}\partial_i\phi + \partial_i f^{i0}\partial_t\phi + \partial_i f^{ij}\partial_j\phi = 0 \ , \tag{5}$$

or changing notation (renaming the label $(t,\mathbf{x})$ by $(\bar{t},\bar{\mathbf{x}})$),

$$-\partial_{\bar{t}}\Phi\partial_{\bar{t}}\phi - \partial_{\bar{t}}V^{\bar{i}}\partial_{\bar{i}}\phi - \partial_{\bar{i}}V^{\bar{i}}\partial_{\bar{t}}\phi + \partial_{\bar{i}}f^{\bar{i}\bar{j}}\partial_{\bar{j}}\phi = 0 \ . \tag{6}$$

A change of coordinates (from $(\bar{t},\bar{\mathbf{x}})$ to $(t,\mathbf{x})$) affects the derivatives in the following way

$$\partial_{\bar{i}} = T^i_{\bar{i}}\partial_i + Z_{\bar{i}}\partial_t \ ; \tag{7}$$

$$\partial_{\bar{t}} = W^i\partial_i + Z\partial_t \ , \tag{8}$$

where

$$T^i_{\bar{i}} := \frac{\partial x^i}{\partial x^{\bar{i}}} \ , \qquad Z_{\bar{i}} := \frac{\partial t}{\partial x^{\bar{i}}} \ , \tag{9}$$

$$W^i := \frac{\partial x^i}{\partial \bar{t}} \ , \qquad Z := \frac{\partial t}{\partial \bar{t}} \ . \tag{10}$$

Let us now proceed term by term with manipulations associated with the transformation of coordinates. We will signal with the symbols $\bar{i}\bar{j})$, $\bar{i}\bar{t})$, etc. the terms containing partial devivatives $\partial_{\bar{i}}\partial_{\bar{j}}$, $\partial_{\bar{i}}\partial_{\bar{t}}$, etc. respectively.

$$\bar{i}\bar{j}) \qquad \partial_{\bar{i}}f^{\bar{i}\bar{j}}\partial_{\bar{j}}\phi = Z_{\bar{i}}\partial_t f^{\bar{i}\bar{j}}T^j_{\bar{j}}\partial_j\phi + T^i_{\bar{i}}\partial_i f^{\bar{i}\bar{j}}Z_{\bar{j}}\partial_t\phi + Z_{\bar{i}}\partial_t f^{\bar{i}\bar{j}}Z_{\bar{j}}\partial_t\phi + T^i_{\bar{i}}\partial_i f^{\bar{i}\bar{j}}T^j_{\bar{j}}\partial_j\phi \ ; \tag{11}$$

The last term in the previous expression can be rewritten as

$$T^i_{\bar{i}}\partial_i f^{\bar{i}\bar{j}} T^j_{\bar{j}}\partial_j\phi = \nabla_i f^{ij}\nabla_j\phi = \frac{1}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}f^{ij}\partial_j\phi \ , \tag{12}$$

where we have introduced a spatial metric γ_{ij}, which is the Euclidean metric $\delta_{\bar{i}\bar{j}}$ written in arbitrary spatial coordinates.

$$\begin{aligned} \bar{t}\bar{i}) \qquad \partial_{\bar{t}}V^{\bar{i}}\partial_{\bar{i}}\phi &= W^i\partial_i V^{\bar{i}} T^j_{\bar{i}}\partial_j\phi + Z\partial_t V^{\bar{i}} T^j_{\bar{i}}\partial_j\phi \\ &\quad + W^i\partial_i V^{\bar{i}} Z_{\bar{i}}\partial_t\phi + Z\partial_t V^{\bar{i}} Z_{\bar{i}}\partial_t\phi \\ &= W^i\partial_i V^j\partial_j\phi + Z\partial_t V^j\partial_j\phi \\ &\quad + W^i\partial_i V^{\bar{i}} Z_{\bar{i}}\partial_t\phi + Z\partial_t V^{\bar{i}} Z_{\bar{i}}\partial_t\phi \ . \end{aligned} \tag{13}$$

$$\begin{aligned} \bar{i}\bar{t}) \qquad \partial_{\bar{i}}V^{\bar{i}}\partial_{\bar{t}}\phi &= T^i_{\bar{i}}\partial_i V^{\bar{i}} W^j\partial_j\phi + T^i_{\bar{i}}\partial_i V^{\bar{i}} Z\partial_t\phi \\ &\quad + Z_{\bar{i}}\partial_t V^{\bar{i}} W^j\partial_j\phi + Z_{\bar{i}}\partial_t V^{\bar{i}} Z\partial_t\phi \ . \end{aligned} \tag{14}$$

$$\begin{aligned} \bar{t}\bar{t}) \qquad \partial_{\bar{t}}\Phi\partial_{\bar{t}}\phi &= Z\partial_t\Phi Z\partial_t\phi + W^i\partial_i\Phi Z\partial_t\phi \\ &\quad + Z\partial_t\Phi W^j\partial_j\phi + W^i\partial_i\Phi W^j\partial_j\phi \ . \end{aligned} \tag{15}$$

Consider now terms of the type ij). From (13)

$$\begin{aligned} W^i\partial_i V^j\partial_j\phi &= \frac{1}{\sqrt{\gamma}}\sqrt{\gamma}W^i\partial_i V^j\partial_j\phi \\ &= \frac{1}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}W^i V^j\partial_j\phi - \frac{1}{\sqrt{\gamma}}\partial_i(\sqrt{\gamma}W^i)V^j\partial_j\phi \ . \end{aligned} \tag{16}$$

If we impose the condition $\nabla_i W^i = 0$, we obtain

$$W^i\partial_i V^j\partial_j\phi = \frac{1}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}W^i V^j\partial_j\phi \ . \tag{17}$$

The same can be done with the last term in (15):

$$W^i\partial_i\Phi W^j\partial_j\phi = \frac{1}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}W^i\Phi W^j\partial_j\phi \ . \tag{18}$$

From the first term in (14) we have

$$T^i_{\bar{i}}\partial_i V^{\bar{i}} W^j\partial_j\phi = \frac{1}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}V^i W^j\partial_j\phi \ . \tag{19}$$

There are several terms of the form it). Considering again the condition $\nabla_i W^i = 0$, we have from (15) and (14) respectively:

$$W^i\partial_i\Phi Z\partial_t\phi = \frac{1}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}W^i\Phi Z\partial_t\phi \ ; \tag{20}$$

$$T^i_{\bar{i}}\partial_i V^{\bar{i}} Z\partial_t\phi = \frac{1}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}V^{\bar{i}}Z\partial_t\phi \ . \tag{21}$$

The terms of the form ti) are

$$Z\partial_t\Phi W^j\partial_j\phi \tag{22}$$

from (15), and

$$Z\partial_t V^j\partial_j\phi \tag{23}$$

from (13). If we now impose $\partial_t\sqrt{\gamma} = 0$, we can rewrite them as

$$\frac{1}{\sqrt{\gamma}}Z\partial_t\sqrt{\gamma}\Phi W^j\partial_j\phi \ ; \tag{24}$$

$$\frac{1}{\sqrt{\gamma}}Z\partial_t\sqrt{\gamma}V^j\partial_j\phi \ . \tag{25}$$

Notice that if we alternatively impose $\partial_i\sqrt{\gamma} = 0$, then all the $\sqrt{\gamma}$ terms in the previous equations disappear, so that we do not need to additionally impose $\partial_t\sqrt{\gamma} = 0$ to recover the initial acoustic form.

If we impose now the condition $\partial_t Z_{\bar{i}} = 0$, we can rewrite the corresponding terms in (11) as

$$\begin{aligned}
&Z_{\bar{i}}\partial_t f^{\bar{i}\bar{j}} T^j_{\bar{j}}\partial_j\phi + Z_{\bar{i}}\partial_t f^{\bar{i}\bar{j}} Z_{\bar{j}}\partial_t\phi \\
&= \partial_t Z_i f^{ij}\partial_j\phi + \partial_t Z_i f^{ij} Z_j\partial_t\phi \\
&= \frac{1}{\sqrt{\gamma}}\partial_t\sqrt{\gamma}Z_i f^{ij}\partial_j\phi + \frac{1}{\sqrt{\gamma}}\partial_t\sqrt{\gamma}Z_i f^{ij} Z_j\partial_t\phi \ .
\end{aligned} \tag{26}$$

If we also impose the condition $\partial_i Z = 0$, then we can rewrite (18):

$$\frac{1}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}W^i V^j\partial_j\phi = \frac{Z}{\sqrt{\gamma}}\partial_i Z^{-1}\sqrt{\gamma}W^i V^j\partial_j\phi \ , \tag{27}$$

and equivalently other similar terms

$$\frac{1}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}W^i\Phi Z\partial_t\phi = \frac{Z}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}W^i\Phi\partial_t\phi \ ; \tag{28}$$

$$T^i_{\bar{i}}\partial_i f^{\bar{i}\bar{j}} Z_{\bar{j}}\partial_t\phi = \frac{Z}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}Z^{-1} f^{ij} Z_j\partial_t\phi \ ; \tag{29}$$

$$W^i\partial_i V^{\bar{i}} Z_{\bar{i}}\partial_t\phi = \frac{Z}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}Z^{-1}W^i V^i Z_i\partial_t\phi \ . \tag{30}$$

To be able to rewrite the terms

$$\frac{1}{\sqrt{\gamma}}\partial_t\sqrt{\gamma}Z_i f^{ij}\partial_j\phi + \frac{1}{\sqrt{\gamma}}\partial_t\sqrt{\gamma}Z_i f^{ij}Z_j\partial_t\phi \ , \tag{31}$$

from (11) as

$$\frac{Z}{\sqrt{\gamma}}\partial_t\sqrt{\gamma}Z^{-1}Z_i f^{ij}\partial_j\phi + \frac{Z}{\sqrt{\gamma}}\partial_t\sqrt{\gamma}Z^{-1}Z_i f^{ij}Z_j\partial_t\phi \ , \tag{32}$$

one needs $\partial_t Z = 0$. However, notice that these terms will not exist if $Z_i = 0$.

Finally, looking at all these conditions, we can conclude that to maintain the form of the acoustic equation, one of the following mutually exclusive sets of conditions is required (all the conditions of a set must hold simultaneously):

- $W_i = 0, \quad Z_{\bar{i}} = 0.$ (33)

- $W_i \neq 0, \quad Z_{\bar{i}} = 0, \quad \nabla_i W^i = 0, \quad \partial_t Z = 0$ or $\partial_i Z = 0, \quad \partial_t\sqrt{\gamma} = 0$ or $\partial_i\sqrt{\gamma} = 0.$ (34)

- $W_i = 0, \quad Z_{\bar{i}} \neq 0, \quad \partial_t Z_{\bar{i}} = 0, \quad \partial_t Z = 0, \quad \partial_t\sqrt{\gamma} = 0$ or $\partial_i\sqrt{\gamma} = 0.$ (35)

- $W_i \neq 0, \quad Z_{\bar{i}} \neq 0, \quad \nabla_i W^i = 0, \quad \partial_t Z_{\bar{i}} = 0, \quad \partial_t Z = 0, \quad \partial_t\sqrt{\gamma} = 0$ or $\partial_i\sqrt{\gamma} = 0.$ (36)

For example, from $Z_{\bar{i}} = 0$ we directly obtain transformations of the form

$$t = f(\bar{t}) \ . \tag{37}$$

Alternatively, with $Z_{\bar{i}} \neq 0$, $\partial_t Z = 0 = \partial_i Z$, we obtain transformations of the form

$$t = C\bar{t} + f(\bar{\mathbf{x}}) \ . \tag{38}$$

In both cases, the space transformations

$$\mathbf{x} = f(\bar{t}, \bar{\mathbf{x}}), \tag{39}$$

have to be such that $\nabla_i W^i = 0$, $\partial_t\sqrt{\gamma} = 0$ or $\partial_i\sqrt{\gamma} = 0$. We can check that a Galilean transformation

$$t = f(\bar{t}) \ ; \qquad \mathbf{x} = \bar{\mathbf{x}} + \mathbf{v}\bar{t} \tag{40}$$

satisfies the conditions. However, the frequency converter transformation described in section 2.2 does not satisfy $\partial_i Z = 0$. A contracting transformation of the form $\mathbf{x} = f(\bar{t})\bar{\mathbf{x}}$ does not satisfy the condition $\nabla_i W^i = 0$.

In the case with more free parameters (Eq. (36)) the new equation can be written as

$$\frac{Z}{\sqrt{\gamma}}\left(-\partial_t\tilde{\Phi}\partial_t\phi - \partial_t\tilde{V}^i\partial_i\phi - \partial_i\tilde{V}^i\partial_t\phi + \partial_i\tilde{f}^{ij}\partial_j\phi\right) = 0 \ , \tag{41}$$

with

$$\tilde{\Phi} = \sqrt{\gamma}\left(\Phi Z - Z_i f^{ij} Z_j Z^{-1} + 2V^i Z_i\right) \ ; \tag{42}$$

$$\tilde{V^i} = \sqrt{\gamma}\left(V^i + W^i\Phi - f^{ij} Z_j Z^{-1} + W^i V^j Z_j Z^{-1}\right) \ ; \tag{43}$$

$$\tilde{f^{ij}} = \sqrt{\gamma} Z^{-1}\left(f^{ij} - W^i V^j - V^i W^j - \Phi W^i W^j\right) \ . \tag{44}$$

Multipliying by an appropriate constant, one will be able to write the tranformed equation (41) in the initial acoustic form (4).

It is clear that the previous form-preserving conditions impose strong restrictions when it comes to mixing time with space. In fact, even a simple transformation such as a space-dependent linear time dilation does not belong to the kind of form-preserving mappings.

We can conclude that the standard transformational approach (based on a direct transformation of the equations) applied to either Eq. (2) or Eq. (3) does not allow us to work with most spacetime transformations. Therefore, another method is required. This is the ATA method, which is summarized in the next section.

2.1.2 Review of ATA basics

The extension of transformation acoustics to general spacetime transformations presents two separate problems. On the one hand, the acoustic equations are not form invariant under transformations that mix space and time (see previous section). As mentioned above, this drawback can be circumvented with the aid of an auxiliary relativistic spacetime. On the other hand, as we will see in the next section, the acoustic systems usually considered in transformation acoustics (which deal with the propagation of acoustic waves in stationary or non-moving fluids) do not posses enough degrees of freedom so as to mimic an arbitrary spacetime transformation. This is the case of the system represented by the standard pressure wave equation, Eq. (2) [20]. The limitations of transformational pressure acoustics have also been analyzed by other authors [38].

Instead of the pressure wave equation, ATA uses the wave equation for the velocity potential, Eq. (3). There are two reasons behind the choice of this equation. First, although it is not form invariant under general spacetime transformations (neither is the pressure wave equation), there is a well-known relativistic model that is analogue to this equation [35, 36]. Second, Eq. (3) allows us to consider the propagation of waves in a moving fluid, which provides the missing degrees of freedom.

From the results of the previous section we know that, if we directly applied a coordinate transformation mixing space and time to Eq. (3), there would appear new terms that could not be ascribed to any property of the medium. The ATA method starts by momentarily interpreting this equation as a different one with better transformation properties. In particular, we use the fact that Eq. (3) can be written as the massless Klein-Gordon equation of a scalar field ϕ_1 propagating in a (3+1)-dimensional pseudo–Riemannian manifold (the abstract spacetime) [35, 36]:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu\phi_1\right) = 0, \tag{45}$$

where $g_{\mu\nu}$ is the 4-dimensional metric (with g its determinant) of the abstract spacetime. The inverse metric $g^{\mu\nu}$ is given by

$$g^{\mu\nu} = \frac{1}{\rho_{\rm V} c_{\rm V}} \left(\begin{array}{ccc} -1 & \vdots & -v_{\rm V}^i \\ \cdots & . & \cdots\cdots\cdots \\ -v_{\rm V}^i & \vdots & c_{\rm V}^2 \tilde{\gamma}^{ij} - v_{\rm V}^i v_{\rm V}^j \end{array} \right). \tag{46}$$

where where for later use the density, the sound speed and the components of background velocity of the fluid have been called $\rho_{\rm V}, c_{\rm V}, v_{\rm V}^i$. We must stress that, although Eqs. (3) and (45) are formally identical when expressed in, for instance, a Cartesian coordinate system, they are completely different equations due to the contrasting nature of the elements that appear in each one. While ρ and c are scalars and $\mathbf{v} = (v_{\rm V}^i)$ is a three-dimensional vector, $g_{\mu\nu}$ is a four-dimensional tensor. Thus, each equation behaves differently under general transformations. The point in interpreting momentarily Eq. (3) as Eq. (45) is that the latter preserves its form under any transformation, including the ones that mix space and time.

The next step is to apply the desired transformation $\bar{x}^\mu = f(x^\mu)$ to Eq. (45). This just implies replacing the metric $g^{\mu\nu}$ by another one $\bar{g}^{\mu\nu}$, which can be derived from the former by using standard tensorial transformation rules

$$\bar{g}^{\bar{\mu}\bar{\nu}} = \Lambda_\mu^{\bar{\mu}} \Lambda_\nu^{\bar{\nu}} g^{\mu\nu}, \tag{47}$$

where $\Lambda_\mu^{\bar{\mu}} = \partial \bar{x}^{\bar{\mu}} / \partial x^\mu$. This way we obtain the transformed version of Eq. (45)

$$\frac{1}{\sqrt{-\bar{g}}} \partial_\mu \left(\sqrt{-\bar{g}}\, \bar{g}^{\mu\nu}\, \partial_\nu \bar{\phi}_1 \right) = 0. \tag{48}$$

Finally, if we relabel the coordinates $\bar{x}^\mu$ of Eq. (48) to x^μ, we can interpret Eq. (48) as the velocity potential wave equation (expressed in the same coordinate system as Eq. (3), e.g., Cartesian) associated with a medium characterized by acoustic parameters $\mathbf{v}_{\rm R}$, $\rho_{\rm R}$ and $c_{\rm R}$

$$-\partial_t \left(\rho_{\rm R} c_{\rm R}{}^{-2} \left(\partial_t \bar{\phi}_1 + \mathbf{v}_{\rm R} \cdot \nabla \bar{\phi}_1 \right) \right) + \nabla \cdot \left(\rho_{\rm R} \nabla \bar{\phi}_1 - \rho_{\rm R} c_{\rm R}^{-2} \left(\partial_t \bar{\phi}_1 + \mathbf{v}_{\rm R} \cdot \nabla \bar{\phi}_1 \right) \mathbf{v}_{\rm R} \right) = 0, \tag{49}$$

which will represent physical space. That is, we identify

$$\bar{g}^{\mu\nu} = \frac{1}{\rho_{\rm R} c_{\rm R}} \left(\begin{array}{ccc} -1 & \vdots & -v_{\rm R}^i \\ \cdots & . & \cdots\cdots\cdots \\ -v_{\rm R}^i & \vdots & c_{\rm R}^2 \tilde{\gamma}^{ij} - v_{\rm R}^i v_{\rm R}^j \end{array} \right). \tag{50}$$

This last equality establishes a relation between the acoustic parameters of virtual and physical space, as $\bar{g}^{\mu\nu}$ is also a function of $\mathbf{v}_{\rm V}$, $\rho_{\rm V}$ and $c_{\rm V}$ (remember that it comes from $g^{\mu\nu}$). As mentioned above, Eqs. (3) and (45) are analogous (formally, they have the same mathematical solutions), and the same happens with Eqs. (48) and (49). These connections guarantee that the velocity potentials ϕ_1 and $\bar{\phi}_1$ in virtual and physical spaces are related by the applied transformation, since so are the fields ϕ_1 and $\bar{\phi}_1$ that appear in Eqs. (45) and (48).

2.1.3 ATA with the pressure wave equation

In transformation acoustics there are two logically separate issues that should not be confused. One issue is whether one uses a pressure equation or a velocity potential equation. Another issue is whether one uses or not an intermediary abstract spacetime to perform the transformation. These two issues are combined in ATA as proposed so far only because the velocity potential equation is the one typically used in acoustics analogue gravity.

Thus, one might wonder whether the ATA method would also work if Eq. (2) was used instead of Eq. (3). Let us show why this is not the case. As in the previous section, to construct such a method, we just need to obtain a connection between the original (not generally form-invariant) laboratory equation, in this case the pressure equation (2), and the relativistic equation (45). Identifying p_1 and ϕ_1 as analogue quantities, these two equations are mathematically identical when the metric $g_{\mu\nu}$ satisfies

$$g^{\mu\nu} = \left(\gamma \det\left(\rho^{ij}\right) B^{-1}\right)^{-\frac{1}{2}} \begin{pmatrix} -B^{-1} & \vdots & 0 \\ \cdots & . & \cdots\cdots\cdots \\ 0 & \vdots & \rho^{ij} \end{pmatrix}, \tag{51}$$

$$g = \det\left(g_{\mu\nu}\right) = -\gamma^2 \det\left(\rho^{ij}\right) B^{-1}. \tag{52}$$

This result can be easily proven. Indeed, knowing that $g^{0i} = g^{i0} = 0$, Eq. (45) becomes

$$\ddot{p}_1 = \frac{-1}{\sqrt{-g}g^{00}} \left(\sqrt{-g} g^{ij} p_{1,j}\right)_{,i}. \tag{53}$$

Substituting the values of $g^{\mu\nu}$ and g given by Eq. (51) and (52) into Eq. (53), we obtain

$$\ddot{p}_1 = \frac{\left(\gamma \det\left(\rho^{ij}\right) B\right)^{1/2}}{\gamma \left(\det\left(\rho^{ij}\right) B^{-1}\right)^{1/2}} \left(\frac{\gamma \left(\det\left(\rho^{ij}\right) B^{-1}\right)^{1/2}}{\left(\gamma \det\left(\rho^{ij}\right) B^{-1}\right)^{1/2}} \rho^{ij} p_{1,j}\right)_{,i}. \tag{54}$$

After simplification,

$$\ddot{p}_1 = \frac{B}{\sqrt{\gamma}} \left(\sqrt{\gamma} \rho^{ij} p_{1,j}\right)_{,i}, \tag{55}$$

i.e. Eq. (2). Therefore, we could use Eq. (2) in laboratory space and employ this analogy between Eqs. (2) and (45). We would start from Eq. (2) particularized for a virtual medium characterized by parameters B_{V} and ρ_{V}^{ij} and obtain its analogue model with an associated metric $g_{\mu\nu}$ given by Eq. (51) particularized for the mentioned parameters. Then, apply the desired transformation to obtain the transformed metric $\bar{g}_{\mu\nu}$.

However, for a transformation that mixes space and time, the required metric components $\bar{g}^{0i}$ and $\bar{g}^{i0}$ will be non-vanishing in general. But according to Eq. (51), the metric associated with the medium in real space must have the following form

$$\tilde{g}^{\mu\nu} = \left(\gamma \det\left(\rho^{ij}\right) B_{\mathrm{R}}^{-1}\right)^{-\frac{1}{2}} \begin{pmatrix} -B_{\mathrm{R}}^{-1} & \vdots & 0 \\ \cdots & . & \cdots\cdots\cdots \\ 0 & \vdots & \rho_{\mathrm{R}}^{ij} \end{pmatrix}. \tag{56}$$

Clearly, then, the ATA condition $\tilde{g}^{\mu\nu} = \bar{g}^{\mu\nu}$ cannot be fulfilled, since $\tilde{g}^{0i} = \tilde{g}^{i0} = 0$. This is why a general spacetime transformation cannot be implemented with an acoustic system described by Eq. (2). On the contrary, the system described by Eq. (3) has an equivalent metric with non-vanishing components $\tilde{g}^{0i}$ and $\tilde{g}^{i0}$. Note that these components would be zero if the background velocity were also zero. Thus, allowing the background fluid to move is a crucial ingredient of the analogue transformation method. However, to our knowledge, there is no similar wave equation to Eq. (3) for the pressure. This shows the importance of choosing the adequate variable to construct a complete transformation approach in this case.

The velocity potential equation by itself needs other assumptions for its validity (in particular, contrarily to the pressure wave equation [39], it does not need a vanishing background pressure gradient – see also the analysis in section 2.3.2 below). In addition, it encompasses more configurations, first by explicitly incorporating background fluid flows, and second by allowing density gradients even with a homogeneous (non-space-dependent) equation of state. Moreover, historically it has been the natural starting point used in Analogue Gravity, while we have just seen that, although a similar analogue metric could be constructed starting from the pressure equation, this would not provide space-time mixing coefficients in the metric. For these reasons (see also Ref. [31]), we use the name ATA explicitly for the combined use of the velocity potential equation and the analogue transformation philosophy.

2.2 Novel devices based on ATA

In this section we study new applications of the ATA method. First, we focus on building the acoustic counterpart of one of the most interesting electromagnetic devices based on spacetime transformations: a frequency shifter. As a second novel application of ATA, we propose a device able to simultaneously compress space and time, which exhibits several interesting features. Finally, we present an acoustic spacetime cloak, a device which permits to conceal a certain set of spacetime events during a limited time interval.

In all three cases, we will need the relation between the parameters of real and virtual media for a transformation of the form ($\bar{t} = f_1(x,t)$; $\bar{x} = f_2(x,t)$), which can be obtained following the ATA method described in section 2.1.2 and reads [31]:

$$v_{\rm R}^x = \left.\frac{\partial_t f_1 \partial_t f_2 - c_{\rm V}^2 \partial_x f_1 \partial_x f_2}{(\partial_t f_1)^2 - c_{\rm V}^2 (\partial_x f_1)^2}\right|_{\bar{x},\bar{t}\to x,t}, \tag{57}$$

$$c_{\rm R}^2 = (v_{\rm R}^x)^2 + \left.\frac{c_{\rm V}^2 (\partial_x f_2)^2 - (\partial_t f_2)^2}{(\partial_t f_1)^2 - c_{\rm V}^2 (\partial_x f_1)^2}\right|_{\bar{x},\bar{t}\to x,t}, \tag{58}$$

$$\rho_{\rm R} = \rho_{\rm V} \frac{c_{\rm R}^2}{c_{\rm V}^2} \left.\frac{(\partial_t f_1)^2 - c_{\rm V}^2 (\partial_x f_1)^2}{\partial_t f_1 \partial_x f_2 - \partial_x f_1 \partial_t f_2}\right|_{\bar{x},\bar{t}\to x,t}. \tag{59}$$

where $\bar{x},\bar{t} \to x,t$ means relabeling $\bar{x},\bar{t}$ to x,t.

2.2.1 Frequency shifter

Here, we will demonstrate that an acoustic frequency converter can be designed with ATA. This was not the case with STA, and therefore illustrates the strength of ATA (see also [31] for other examples of applications which can be designed with ATA but not with STA).

Let us consider the following transformation:

$$\bar{t} = t(1 + ax), \tag{60}$$

$$\bar{x}^i = x^i, \tag{61}$$

with a having units of inverse length. This is an interesting transformation that has been used in the context of transformation optics to design frequency converters [30]. Obviously, this transformation does not satisfy the form-invariance conditions of STA, as it mixes space and time. As a consequence, the use of the ATA approach is indispensable in this case.

According to Eqs. (57)–(59), the transformation in Eq. (60) can be implemented by using the following parameters:

$$v_{\rm R}^x = \frac{-c_{\rm V}^2 at(1+ax)}{(1+ax)^4 - (c_{\rm V}at)^2}, \tag{62}$$

$$\frac{c_{\rm R}}{c_{\rm V}} = \frac{\rho_{\rm R}}{\rho_{\rm V}} = \frac{(1+ax)^3}{(1+ax)^4 - (c_{\rm V}at)^2}. \tag{63}$$

In a practical situation, this transformation is only applied in a certain region $0 < x < L$. Taking $a = (m^{-1} - 1)/L$, with m a constant, we ensure that the transformation is continuous at $x = 0$. In this case, $\bar{t} = t/m$ at $x = L$. Continuity can be guaranteed at $x = L$ by placing the medium that implements the transformation ($\bar{t} = t/m$; $\bar{x}^i = x^i$) at the device output $(x > L)$. It can easily be shown that the properties of such a medium are

$$v_{\rm R}^x = 0\,; \qquad \frac{c_{\rm R}}{c_{\rm V}} = \frac{\rho_{\rm R}}{\rho_{\rm V}} = m. \tag{64}$$

Following a reasoning similar to that of Ref. [30], it is possible to prove that, after going through the device, the acoustic signal frequency is scaled by a factor of m. To that end, we assume that the potential has the form of a plane wave with a space-dependent frequency $\phi_1 = \phi_c \exp\left(i\omega(x)t - ikx\right)$ (the problem is invariant in the y and z directions). Substituting this *ansatz* into the wave equation and neglecting $\partial_x^2\omega$, we obtain the following relation

$$-\omega c^{-2}\left(\omega + v_x\Delta_x\right) + \Delta_x\left[\Delta_x - c^{-2}\left(w + v_x\Delta_x\right)\right] = 0, \tag{65}$$

with $\Delta_x = td\omega/dx - k$. Inserting Eqs. (62)–(63) into Eq. (65), we arrive at

$$\left[\frac{\omega\left(1+ax\right)}{c_{\rm V}}\right]^2 = \left[\Delta_x + \frac{\omega at}{1+ax}\right]^2. \tag{66}$$

The solution to this equation is the following dispersion relation

$$\omega(x) = \pm \frac{c_{\mathrm{V}} k}{1 + ax}, \tag{67}$$

from which it is clear that $\omega(x = L) = m\omega(x = 0)$.

Note that, as in the optical case, a time-varying medium is required to achieve a frequency conversion, while in static media only a wavelength change can be observed (as a consequence of the medium spatial variation), as for instance in the case analyzed in [40].

In order to check the validity of the acoustic frequency converter, a particular case with $L = 5$ m and $m = 0.8$ was simulated numerically. The calculation was carried out by using the acoustic module of COMSOL Multiphysics, where the weak form of the aeroacoustic wave equation was modified to allow for time-varying density and speed of sound. In the simulation, an acoustic wave impinges onto the frequency converter from the left. The space dependence of the velocity potential at a certain instant is depicted in Fig. 1(a). It can be seen that, while the wavelength grows with x inside the converter, it is the same at the input and output media. Since the speed of sound of the output medium is m times that of the input medium, the output frequency must also be m times the input one. This relation can also be obtained by comparing the time evolution of the velocity potential at two arbitrary positions to the left and right of the converter [see Fig. 1(b)]. To further verify the functionality of the converter, we calculated the trajectory followed by an acoustic ray immersed in such a medium, starting at $x = 0, t = 0$. This was done by solving numerically Hamilton's equations (see Ref. [31] for further details). In this case we chose $a = 1$ m^{-1}. The calculated time-position curve and the expected theoretical curve are depicted in Fig. 1(c). Both are identical. For comparison purposes, the curve associated to the propagation of sound in the reference fluid is also shown.

2.2.2 Spacetime compressor

The very first application of ATA, a device that dynamically compresses space without changing the time variable, was designed previously in [31]. In practice, this compression can only be performed inside a box. As a consequence, we created a discontinuity at the box boundaries that could produce reflections. To avoid these reflections, we had to ensure that there is no compression at the instant rays enter or exit the box.

In this section, we analyze the possibility of performing a continuous transformation of both space and time that eliminates this problem. Specifically, we will consider the mapping ($\bar{x} = x\bar{r}/r$; $\bar{t} = t\bar{r}/r$), with

$$r = \sqrt{x^2 + (c_{\mathrm{V}} t)^2} \tag{68}$$

and

$$\bar{r} = \begin{cases} \frac{r_1}{r_2} r, & 0 \le r \le r_2 \\ ar - b, & r_2 < r \le r_3 \\ r, & r > r_3 \end{cases} \tag{69}$$

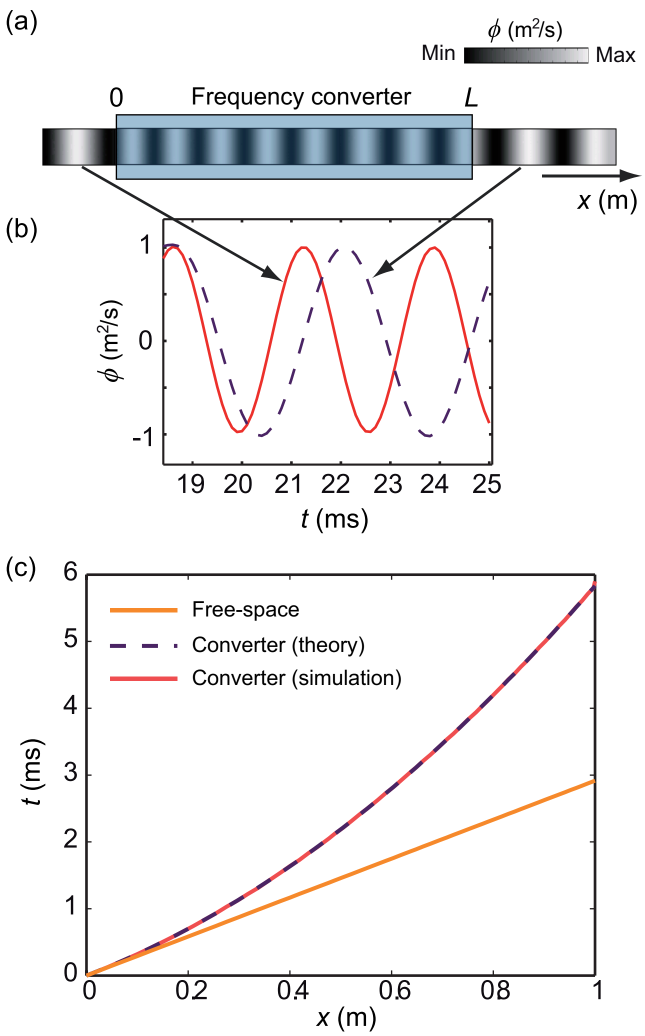

Figure 1: Acoustic frequency converter designed with ATA. (a) Velocity potential as a function of space at a given instant. (b) Time dependence of the velocity potential at two different positions to the left and right of the converter. (c) Trajectory followed by an acoustic ray inside a converter with $a = 1\ \mathrm{m}^{-1}$.

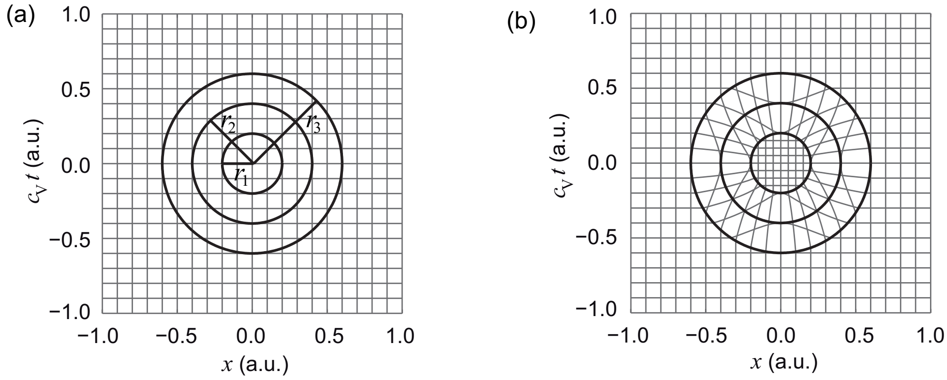

Figure 2: Transformation that simultaneously compresses space and time. (a) Cartesian grid in virtual space. (b) Grid deformed by the proposed transformation.

where $a = (r_3 - r_1)/(r_3 - r_2)$ and $b = (r_2 - r_1)/(r_3 - r_2)r_3$. Basically, the circle $r = r_2$ is squeezed into the circle $r = r_1$. To guarantee the continuity of the transformation, the annular region $r_2 < r < r_3$ is expanded to the region $r_1 < r < r_3$ (see Fig. 2). The transformation is similar to the one that was employed to compress a region of two-dimensional space [41], but it is now applied to compress a region of two-dimensional spacetime (time and one space variable), which significantly changes its meaning. Here we will limit the analysis to the propagation of rays, although it could be extended easily to the propagation of waves.

Particularizing Eqs. (57)–(59) to the desired compressing transformation we are led to the sought parameters for the region $r_1 < r < r_3$ (note that ρ_R is not relevant in ray acoustics)

$$v_R^x = c_V \frac{c_V b^2 x t (c_V^2 t^2 - x^2)}{r^6 + 2bc_V^2 t^2 r^3 + (bc_V t)^2 (c_V^2 t^2 - x^2)}, \quad (70)$$

$$c_R = c_V \frac{r^3 (r^3 + br^2)}{r^6 + 2bc_V^2 t^2 r^3 + (bc_V t)^2 (c_V^2 t^2 - x^2)}. \quad (71)$$

Interestingly, for the inner disk ($r < r_1$), we find that $c_R = c_V$ and $\mathbf{v}_R = \mathbf{v}_V = \mathbf{0}$, i.e., we do not need to change the background medium in this region. This is in contrast with the devices that implement a uniform compression of space, whose refractive index changes in proportion to the compression factor [41, 42]. The reason is that, in a uniformly compressed spatial region, rays have to travel a shorter distance in the same time, and thus the propagation speed must decrease (the refractive index increases). However, in the case studied here time is also compressed, and rays have to travel this distance in a shorter time as well. Another way to see it is by noticing that, according to the transformation, trajectories have the same slope in the regions $r < r_1$ and $r > r_3$, which implies that they propagate at the same speed (same refractive index). This could be interesting for applications where we want to have the compressed wave directly in the background medium (in this case in the region $r < r_1$, since the compressor is limited to the ring $r_1 < r < r_3$).

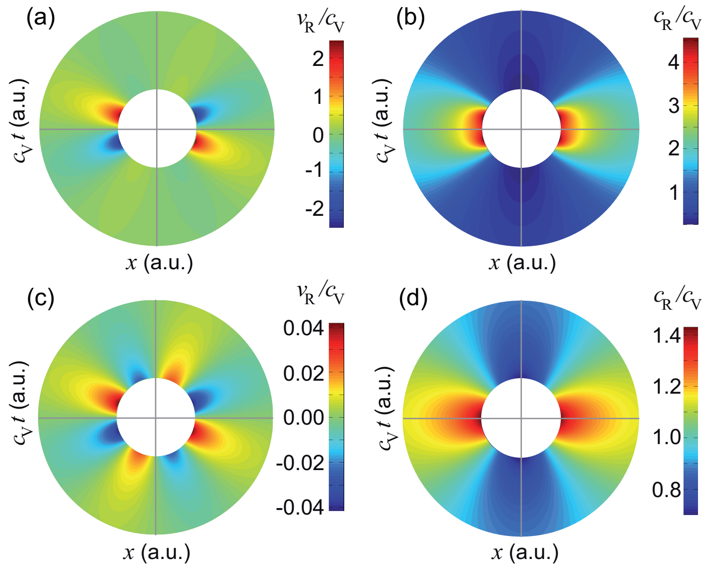

Figure 3: Distribution of the background velocity and speed of sound required for the implementation of a space-time compressor with $r_1 = 0.2$ and $r_3 = 0.6$ for two different values of r_2. (a-b) $r_2 = 0.4$. (c-d) $r_2 = 0.25$.

Another remarkable feature of this compressor is that the frequency of the acoustic wave is increased within the disk $r < r_1$ by the same factor as the wavelength decreases (an additional way to explain why the refractive index is not modified there), since time and space are equally compressed. Therefore, we also achieve a frequency shift in an area filled with the background medium, something that does not occur in other transformation-based frequency converters in which only time is transformed [20, 30].

The distribution of background velocity and speed of sound required for the implementation of the space-time compressor is shown in Fig. 3 for two different values of r_2. From a technological point of view, both parameters take a set of reasonable values at any given instant (note that the parameters become more relaxed as the compression factor decreases, i.e., as r_2 approaches r_1). However, the real challenge comes from the fact that the medium properties must change in time. Recent works have demonstrated different ways of achieving a dynamic control of the speed of sound that could be employed for the implementation of ATA devices [43, 44]. It would just remain to solve the problem of attaining the desired distribution of background velocities. In the optical case, a moving medium can be mimicked by using background waves able to modify dynamically the electromagnetic properties of a certain material [45]. In principle, a similar strategy could be followed in our acoustic problem.

To verify the functionality of the spacetime compressor, we simulated the trajectories followed by different rays going through it (see Fig. 4). Again, this was done by solving numerically Hamilton's equations [31]. The calculated trajectories are compared with

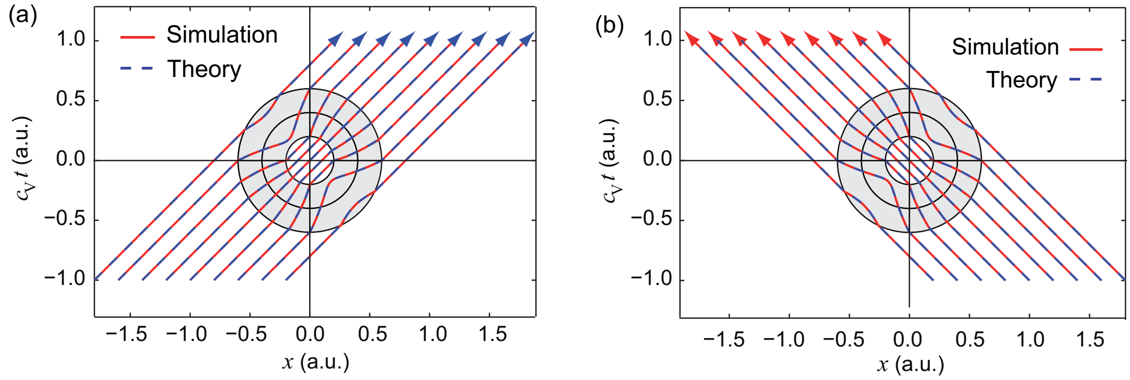

Figure 4: Performance of the proposed spacetime compressor (shaded region). Here, $r_1 = 0.2$, $r_2 = 0.4$, and $r_3 = 0.6$. The device is symmetric and works equally well for rays traveling in (a) the $+x$ direction and (b) the $-x$ direction.

the expected theoretical ones (obtained by applying the proposed transformation to the trajectories of the corresponding rays in virtual space, which are just straight lines), finding an excellent agreement.

Finally, it is worth mentioning that, unlike the time cloaks analyzed in previous studies [28, 31], the proposed compressor works for acoustic waves traveling in both the positive and negative x-direction. This is shown in Fig. 4.

2.2.3 Spacetime cloak

As a third application of ATA, we have designed the acoustic counterpart of the spacetime cloak reported for electromagnetic waves in [28], and recently verified experimentally [29]. Unlike static invisibility cloaks, this device conceals only a certain set of spacetime events occurring during a limited time interval. To show how the proposed method allows for designing an acoustic spacetime cloak, we consider the following example with a single spatial dimension.

The transformation can be directly adapted from the definition given for the optical case in [28]. It consists of the composition of a Lorentz boost

$$x_1 = (1 - n^{-2})^{-1/2} \left(x - \frac{ct}{n} \right), \tag{72}$$

$$t_1 = (1 - n^{-2})^{-1/2} \left(t - \frac{x}{nc} \right), \tag{73}$$

followed by a curtain map

$$x_2 = \left(\frac{\delta + |ct_1|}{\delta + n\sigma} \right) [x_1 - \mathrm{sgn}(x_1)\sigma] + \mathrm{sgn}(x_1)\sigma, \tag{74}$$

$$t_2 = t_1, \tag{75}$$

and an inverse Lorentz boost

$$\bar{x} = (1 - n^{-2})^{-1/2}\left(x_2 + \frac{ct_2}{n}\right), \tag{76}$$

$$\bar{t} = (1 - n^{-2})^{-1/2}\left(t_2 + \frac{x_2}{nc}\right). \tag{77}$$

In our example $\sigma = 1$, $n = 2$ and $\delta = 0.5$.

This transformation mixes time with one spatial variable. Therefore, the material parameters associated with the cloak can be obtained by substituting the mentioned transformation into Eqs. (57)–(59) above. The resulting parameters are shown in Fig. 5. Fig. 5 also compares the numerical simulation of the propagation of a set of acoustic rays through such a medium with the theoretical prediction, finding a near-perfect agreement (see [31] for further details).

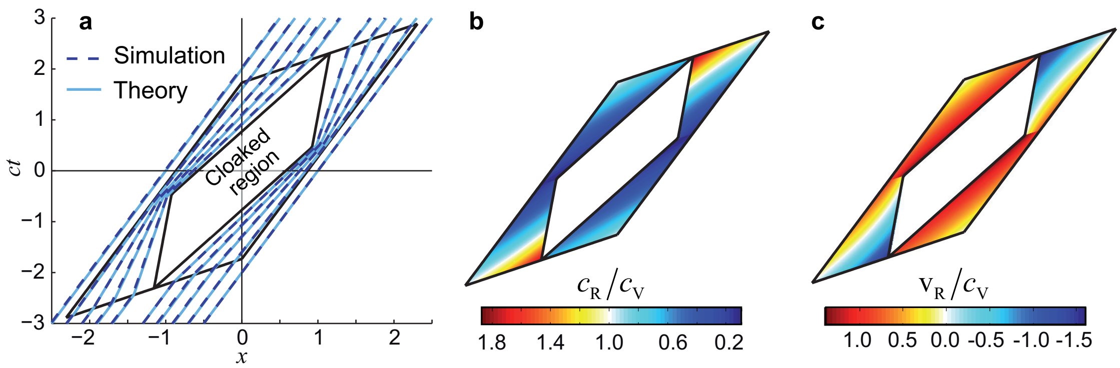

Figure 5: Acoustic spacetime cloak. (a) Simulated acoustic rays. As can be seen, sound propagation is speeded up or slowed down in order to hide any acoustic event belonging to the cloaked region. Rays exit the cloak as if they had just passed through the background fluid. (b) Sound speed and (c) background velocity $\mathbf{v}_R = v_R\hat{x}$ of the cloak.

2.3 Anisotropic transformations with ATA

Like in any other transformation approach, also in the case of ATA the prescribed acoustic parameters are smooth functions of the coordinates and show an anisotropic character. However, the actual construction process of acoustic devices relies on the use of natural materials which only provide a discrete set of isotropic acoustic properties. The problem then arises to connect the theoretical results of ATA and the technological realization of the required acoustic media. To solve this problem we can resort to acoustic metamaterials or metafluids, i.e., fluids made up of various materials with certain acoustic parameters (usually homogeneous and isotropic), which under some given conditions display different effective parameters that depend on those of the constitutive materials and on their shapes [4, 46]. One way to obtain these effective parameters is to use homogenization techniques [47,48]. In the case of periodic systems whose physics can be described by differential equations with oscillating coefficients, the homogenization allows to approximate the full equations with equivalent ones containing homogeneous coefficients. As far as acoustic metamaterials are concerned, these techniques have been widely used to homogenize the pressure wave equation [5,25,49,50]. It is then reasonable to expect that the same approach can be used to achieve a better link between the ATA prescriptions and real systems.

Here, we address specifically such connection. In section 2.3.1, we describe the application of a two-scale homogenization technique to the velocity potential wave equation and present several examples. Surprisingly, we find that the resulting effective parameters are different from those arising from the homogenized pressure equation. In section 2.3.2 we analyze the origin of this behavior. Our analysis reveals new interesting insights on the way in which the velocity potential equation (and thus ATA) works, as well as an alternative way to construct metafluids based on gradients of the static pressure. Finally, in section 2.3.3 we use the results of section 2.3.1 to design a device able to cloak the acoustic velocity potential.

2.3.1 Homogenization of the acoustic equations

Our initial goal is to homogenize the equation for the acoustic perturbation of the velocity potential ϕ_1. For simplicity, we will start by considering a non-moving background and rewrite this equation (we use the notation $\partial_t = \partial/\partial t$ and $\partial_i = \partial/\partial x^i$, with $x^1 = x$, $x^2 = y$ and $x^3 = z$) as

$$-C\partial_t^2\phi_1 + \partial_i\left(a^{ij}\partial_j\phi_1\right) = 0, \tag{78}$$

with $C = \rho_0 c_0^{-2}$ and $a^{ij} = \rho_0\delta^{ij}$, where ρ_0 and c_0 are the static or background mass density and speed of sound, respectively. In addition, we would like to compare the resulting effective parameters with the ones arising from the homogenization of the pressure wave equation. Eq. (78) also represents the pressure wave equation if we take $C = \rho_0^{-1}c_0^{-2}$ and $a^{ij} = \rho_0^{-1}\delta^{ij}$, and replace ϕ_1 by the acoustic pressure p_1. We will assume that the composite to be homogenized is periodic in such a way that $\rho_0(\mathbf{x}) = \rho_0(\mathbf{x}+\mathbf{NY})$ and $c_0(\mathbf{x}) = c_0(\mathbf{x}+\mathbf{NY})$, where $\mathbf{N} = diag(n_1, n_2, n_3)$ is a diagonal matrix with n_i an integer number and $\mathbf{Y} = (Y_1, Y_2, Y_3)^T$ is a constant vector that determines the periodicity in each Cartesian direction. Moreover, we will assume that the coefficients a^{ij} satisfy the ellipticity condition $a^{ij}v_iv_j \geq \alpha|\mathbf{v}|^2$ ($\alpha > 0$) for all $\mathbf{v} \in \mathbb{R}^3$. Under such conditions, we can homogenize this equation via the so-called two-scale approach, which studies the asymptotic behavior of the equation as the medium periodicity tends to zero [47, 48]. The dependence on the microstructure's periodicity is encoded through a parameter ϵ, which is proportional to its length scale. Then, the coefficients of the partial differential equation a^{ij} are expressed as a function of $\mathbf{y} = \mathbf{x}/\epsilon$, while any other function depends both on $\mathbf{x}$ and $\mathbf{y}$. For each value of ϵ we have an equation

$$-C(\mathbf{y})\partial_t^2\phi_1(\mathbf{x},\mathbf{y},t) + \mathcal{A}^\epsilon\phi_1(\mathbf{x},\mathbf{y},t) = 0, \tag{79}$$

where $C(\mathbf{y}) = \rho(\mathbf{y})c^{-2}(\mathbf{y})$ and we have defined the operator $\mathcal{A}^\epsilon$ as

$$\mathcal{A}^\epsilon := \frac{\partial}{\partial x^i}\left(a^{ij}(\mathbf{y})\frac{\partial}{\partial x^j}\right). \tag{80}$$

Using the chain rule, we can express $\mathcal{A}^\epsilon$ as

$$\mathcal{A}^\epsilon = \epsilon^{-2}\mathcal{A}^0 + \epsilon^{-1}\mathcal{A}^1 + \mathcal{A}^2, \tag{81}$$

with

$$\mathcal{A}^0 := \frac{\partial}{\partial y^i}\left(a^{ij}(\mathbf{y})\frac{\partial}{\partial y^j}\right); \tag{82}$$

$$\mathcal{A}^1 := \frac{\partial}{\partial x^i}\left(a^{ij}(\mathbf{y})\frac{\partial}{\partial y^j}\right) + \frac{\partial}{\partial y^i}\left(a^{ij}(\mathbf{y})\frac{\partial}{\partial x^j}\right); \tag{83}$$

$$\mathcal{A}^2 := \frac{\partial}{\partial x^i}\left(a^{ij}(\mathbf{y})\frac{\partial}{\partial x^j}\right). \tag{84}$$

We seek a solution of the form

$$\phi_1(\mathbf{x},\mathbf{y},t) = \phi_1^0(\mathbf{x},\mathbf{y},t) + \epsilon\phi_1^1(\mathbf{x},\mathbf{y},t) + \epsilon^2\phi_1^2(\mathbf{x},\mathbf{y},t), \tag{85}$$

where each function ϕ_1^i is periodic in $\mathbf{y}$. Substituting Eq. (81) and Eq. (85) into Eq. (79) and equating equal powers of ϵ we obtain the following set of equations

$$\mathcal{A}^0\phi_1^0 = 0\ ; \tag{86}$$

$$\mathcal{A}^0\phi_1^1 = -\mathcal{A}^1\phi_1^0\ ; \tag{87}$$

$$\mathcal{A}^0\phi_1^2 = C(\mathbf{y})\partial_t^2\phi_1^0 - \mathcal{A}^1\phi_1^1 - \mathcal{A}^2\phi_1^0\ . \tag{88}$$

Next, we will need the following theorem:

Theorem

Let ϕ be a $\mathbf{Y}$-periodic function. Then, the equation

$$\mathcal{A}^0\phi = F \tag{89}$$

has a (unique) solution (up to a constant) if and only if

$$\langle F\rangle = \frac{1}{|Y|}\int_Y F d\mathbf{y} = 0. \tag{90}$$

The volume of the base cell is denoted by $|Y|$. As a consequence of this theorem, we know that there exists a solution to Eq. (86). Moreover, using the ellipticity condition, it can be shown that Eq. (86) implies that

$$\phi_1^0(\mathbf{x},\mathbf{y}) = \phi_1^0(\mathbf{x}). \tag{91}$$

Proof

From Eq. (86), it follows immediately that

$$\int_Y \phi_1^0 \mathcal{A}^0 \phi_1^0 d\mathbf{y} = 0. \tag{92}$$

On the other hand, integrating by parts over a unit cell we have

$$\int_Y \phi_1^0 \mathcal{A}^0 \phi_1^0 d\mathbf{y} + \int_Y a^{ij} \frac{\partial \phi_1^0}{\partial y_i} \frac{\partial \phi_1^0}{\partial y^j} d\mathbf{y} = \int_Y \frac{\partial}{\partial y_i} \left(\phi_1^0 a^{ij} \frac{\partial \phi_1^0}{\partial y^j} \right) d\mathbf{y} = 0, \tag{93}$$

where the last equality follows from the fact that, if a function $F(\mathbf{y})$ is periodic in y^i, then $\int_Y \frac{\partial F}{\partial y^i} d\mathbf{y} = 0$ due to the fundamental theorem of calculus. Therefore, we have

$$\int_Y a^{ij} \frac{\partial \phi_1^0}{\partial y_i} \frac{\partial \phi_1^0}{\partial y^j} d\mathbf{y} = 0 \tag{94}$$

Finally, the ellipticity condition allows us to write

$$\int_Y a^{ij} \frac{\partial \phi_1^0}{\partial y_i} \frac{\partial \phi_1^0}{\partial y^j} d\mathbf{y} \geq \alpha \int_Y |\nabla_y \phi_1^0|^2 d\mathbf{y}. \tag{95}$$

Fulfillment of the last two equations implies

$$\nabla_y \phi_1^0 = 0 \rightarrow \phi_1^0(\mathbf{x}, \mathbf{y}) = \phi_1^0(\mathbf{x}). \ \blacksquare \tag{96}$$

Using this result in Eq. (91) we obtain

$$\mathcal{A}^1 \phi_1^0 = \frac{\partial}{\partial x^i} \left(a^{ij}(\mathbf{y}) \frac{\partial \phi_1^0(\mathbf{x})}{\partial y^j} \right) + \frac{\partial}{\partial y^i} \left(a^{ij}(\mathbf{y}) \frac{\partial \phi_1^0(\mathbf{x})}{\partial x^j} \right) = \frac{\partial \phi_0(\mathbf{x})}{\partial x^j} \frac{\partial a^{ij}(\mathbf{y})}{\partial y^i}. \tag{97}$$

Thus, $\langle \mathcal{A}_1 \phi_1^0 \rangle = 0$ and Eq. (87) has a unique solution, which we assume to be of the form

$$\phi_1^1(\mathbf{x}, \mathbf{y}, t) = \chi^j(\mathbf{y}) \frac{\partial \phi_1^0(\mathbf{x}, t)}{\partial x^j} + \phi_C(\mathbf{x}). \tag{98}$$

Substitution of Eq. (98) into Eq. (87) gives rise to the so-called *cell problem*

$$\mathcal{A}^0 \chi^k(\mathbf{y}) = -\frac{\partial a^{ik}}{\partial y^i}, \tag{99}$$

which provides the sought functions $\chi^j(\mathbf{y})$. The cell problem is guaranteed to have a unique solution, since

$$\langle \frac{\partial a^{ik}}{\partial y^i} \rangle = 0. \tag{100}$$

Finally, the solvability condition for Eq. (88) reads

$$\int_Y \left(\mathcal{A}^1\phi_1^1 + \mathcal{A}^2\phi_1^0\right) d\mathbf{y} = \langle C(\mathbf{y})\rangle \partial_t^2 \phi_1^0. \tag{101}$$

With the help of the following partial results

$$\int_Y \mathcal{A}^2\phi_1^0 d\mathbf{y} = \frac{\partial^2 \phi_1^0(\mathbf{x})}{\partial x^i \partial x^j} \int_Y a^{ij}(\mathbf{y}) d\mathbf{y}, \tag{102}$$

$$\int_Y \mathcal{A}^1\phi_1^1 d\mathbf{y} = \frac{\partial^2 \phi_1^0(\mathbf{x})}{\partial x^i \partial x^k} \int_Y a^{ij}(\mathbf{y}) \frac{\partial \chi^k(\mathbf{y})}{\partial y^j} d\mathbf{y}, \tag{103}$$

we obtain the *homogenized equation*

$$-\langle C(\mathbf{y})\rangle \partial_t^2 \phi_1^0 + \tilde{a}^{ij} \frac{\partial^2 \phi_1(\mathbf{x})}{\partial x^i \partial x^j} = 0, \tag{104}$$

where the *effective coefficient* $\tilde{a}^{ij}$ is given by

$$\tilde{a}^{ij} = \frac{1}{|Y|} \int_Y \left(a^{ij} + a^{ik} \frac{\partial \chi^j(\mathbf{y})}{\partial y^k} \right) d\mathbf{y}. \tag{105}$$

There exist specific situations for which the cell problem has an analytical solution. Let us analyze two of them.

Case 1

Consider a medium whose properties vary only along the y_1 direction so that the density is of the form

$$a^{ij}(\mathbf{y}) = \rho(y_1)\delta^{ij}. \tag{106}$$

In this case, the cell problem leads to the following equations

$\underline{k=1}$

Assuming that $\chi^1 = \chi^1(y_1)$, the cell problem is

$$\frac{\partial}{\partial y_1}\left(\rho(y_1)\frac{\partial \chi^1(y_1)}{\partial y_1}\right) = -\frac{\partial \rho(y_1)}{\partial y_1}. \tag{107}$$

Integrating over y_1 we obtain

$$\frac{\partial \chi^1(y_1)}{\partial y_1} = -1 + \frac{K_1}{\rho(y_1)}. \tag{108}$$

Integrating again

$$\chi^1(y_1) = -y_1 + K_1 \int_0^{y_1} \frac{1}{\rho(y_1)} dy_1 + K_2, \tag{109}$$

where K_1 and K_2 are constants. The value of K_1 can be determined by using the fact that χ^1 is Y_1-periodic (i.e., $\chi(Y_1) - \chi(0) = 0$), obtaining

$$K_1 = \left(\frac{1}{Y_1} \int_0^{Y_1} \frac{1}{\rho(y_1)} dy_1 \right)^{-1} = \langle \rho(y_1)^{-1} \rangle^{-1}. \tag{110}$$

It is not necessary to calculate K_2, as only the derivatives of χ^1 enter the expression of $\tilde{a}^{ij}$. Moreover, by the theorem above, any other solution will differ from this one by a constant.

$\underline{k = 2}$

We have

$$\mathcal{A}^0 \chi^2((y)) = 0 \rightarrow \chi^2 = \text{constant}. \tag{111}$$

The same result is obtained for $k = 3$. Introducing the calculated functions χ^k into Eq. (105) we find that

$$\begin{aligned} \tilde{a}^{ij} &= \frac{1}{Y_1} \int_{Y_1} \left(\rho(y_1) \delta^{ij} + \rho(y_1) \delta^{ik} \frac{\partial \chi^j(\mathbf{y})}{\partial y_k} \right) dy_1 = \\ &= \langle \rho(y_1) \rangle \delta^{ij} + \left(-\langle \rho(y_1) \rangle + \langle \rho(y_1)^{-1} \rangle^{-1} \right) \delta^{1i} \delta^{1j}. \end{aligned} \tag{112}$$

Therefore,

$$\tilde{a}^{ij} = \begin{pmatrix} \langle \rho(y_1)^{-1} \rangle^{-1} & 0 & 0 \\ 0 & \langle \rho(y_1) \rangle & 0 \\ 0 & 0 & \langle \rho(y_1) \rangle \end{pmatrix}. \tag{113}$$

Case 2

As another example, consider the following set of coefficients

$$a^{ij}(\mathbf{y}) = \rho_1(y_1) \rho_2(y_2) \rho_3(y_3) \delta^{ij}. \tag{114}$$

Then, the cell equations are

$\underline{k = 1}$

$$\frac{\partial}{\partial y_i} \left(\rho(y_1) \rho(y_2) \rho(y_3) \delta^{ij} \frac{\partial \chi^1(y_1)}{\partial y_j} \right) = -\rho(y_2) \rho(y_3) \frac{\partial \rho(y_1)}{\partial y_1}. \tag{115}$$

We try a solution of the form $\chi^1(\mathbf{y}) = \chi^1(y_1)$, in which case the previous equation reduces to

$$\frac{\partial}{\partial y_1} \left(\rho(y_1) \frac{\partial \chi^1(y_1)}{\partial y_1} \right) = -\frac{\partial \rho(y_1)}{\partial y_1}, \tag{116}$$

which was studied in Case 1. A very similar result is obtained for $k = 2,3$. Using Eq. (105), it can be shown that

$$\tilde{a}^{ij} = \begin{pmatrix} \frac{\langle \rho(y_2)\rho(y_3)\rangle}{\langle \rho(y_1)^{-1}\rangle} & 0 & 0 \\ 0 & \frac{\langle \rho(y_1)\rho(y_3)\rangle}{\langle \rho(y_2)^{-1}\rangle} & 0 \\ 0 & 0 & \frac{\langle \rho(y_1)\rho(y_2)\rangle}{\langle \rho(y_3)^{-1}\rangle} \end{pmatrix}. \tag{117}$$

As a consequence of Eq. (105), we can define the following effective anisotropic densities for the homogenized velocity potential and pressure equations,

$$\tilde{\rho}_\phi^{ij} = \left\langle \rho_0 \delta^{ij} + \rho_0 \delta^{ik} \frac{\partial \chi^j(\mathbf{x})}{\partial x^k} \right\rangle; \tag{118}$$

$$(\tilde{\rho}_p^{ij})^{-1} = \left\langle \rho_0^{-1} \delta^{ij} + \rho_0^{-1} \delta^{ik} \frac{\partial \chi^j(\mathbf{x})}{\partial x^k} \right\rangle. \tag{119}$$

Eq. (119) is the usual definition employed in the literature. Note that, although we have considered a simplified problem with a non-moving background, the appearance of an effective anisotropic density in the homogenized ϕ-equation already enables us to extend ATA to anisotropic spatial transformations of the velocity potential (which are physically different from transformations of the pressure [20, 31]). However, there is an issue that deserves further attention; the effective densities arising from the homogenization of the ϕ-equation and the p-equation might be different, as deduced from Eqs. (118) and (119). A similar conclusion can be drawn about the effective speeds of sound.

As an example, let us consider the homogenization of a two-dimensional (three-dimensional) periodic array of cylinders (spheres) of radius r, with constant background parameters ρ_B and c_B embedded in a fluid with parameters ρ_A and c_A. Since the resulting effective media will be isotropic, we can also define the following effective speeds of sound

$$\tilde{c}_\phi^2 = \tilde{\rho}_\phi \langle \rho_0 c_0^{-2} \rangle^{-1}, \tag{120}$$

$$\tilde{c}_p^2 = \tilde{\rho}_p^{-1} \langle \rho_0^{-1} c_0^{-2} \rangle^{-1}, \tag{121}$$

where $\tilde{\rho}_\phi^{ij} = \tilde{\rho}_\phi \delta^{ij}$ and $\tilde{\rho}_p^{ij} = \tilde{\rho}_p \delta^{ij}$. Specifically, let us study a common configuration in which the cylinders (spheres) are made of wood and the surrounding medium is air [49]. Thus, we can take $\rho_B \approx 700\rho_A$ and $c_B \approx 10 c_A$. We calculated the corresponding effective parameters by solving numerically Eq. (99) and Eqs. (118)–(121) in COMSOL Multiphysics. Note that the derivative on the right-hand side of Eq. (99) is not defined at the interface between both media. To deal with this situation we solve the corresponding equation at each uniform domain and apply the proper boundary condition at the discontinuity [25]. Using the divergence theorem, it is not difficult to show that the natural boundary condition associated with Eq. (99) reads

$$\rho_2 \frac{\partial \chi_i}{\partial n}\bigg|_2 - \rho_1 \frac{\partial \chi_i}{\partial n}\bigg|_1 = -(\rho_2 - \rho_1)\mathbf{e}_i \cdot \mathbf{n}, \tag{122}$$

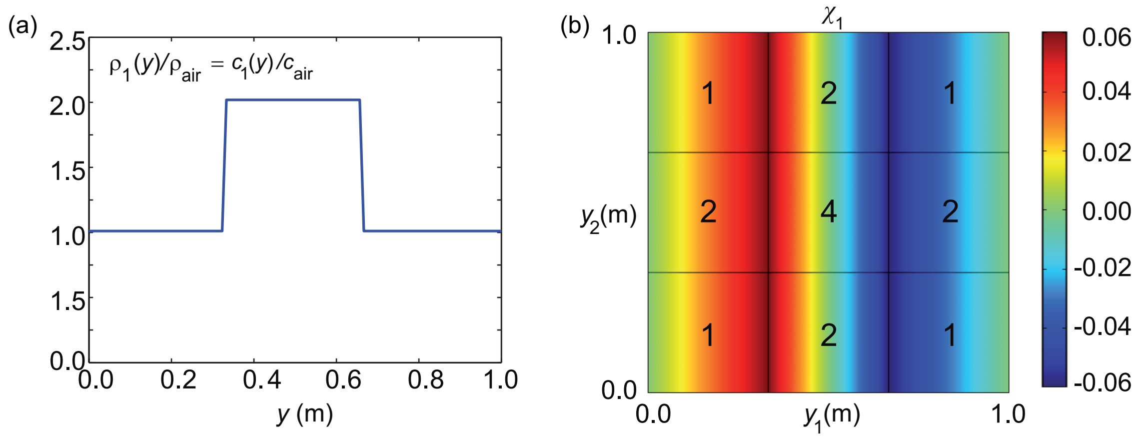

Figure 6: Verification example. (a) Function $\rho_1(y)$ (b) Numerically calculated values of χ_1 for the pressure wave equation and values of density and speed of sound in each region (normalized to those of air).

where the subscripts 1 and 2 refer to each of the two media, $\mathbf{e}_i$ are the basis vectors in Cartesian coordinates, and $\mathbf{n}$ denotes the unit normal to the boundary between both media, pointing from medium 1 to medium 2.

Verification test

To verify the proposed numerical method for obtaining the effective parameters of a given periodic medium, we calculated those of a simple 2D unit cell whose acoustic properties can be expressed as

$$\rho(y_1, y_2) = \rho_1(y_1)\rho_1(y_2), \tag{123}$$

$$c(y_1, y_2) = c_1(y_1)c_1(y_2) = \rho_1(y_1)\rho_1(y_2). \tag{124}$$

The particular values considered for $\rho_1(y)$ are shown in Fig.6a. This corresponds to a medium with 9 homogeneous regions (see Fig. 6b). The values of density and speed of sound (ρ/ρ_{air} and c/c_{air}) in each region (normalized to those of air, which is taken as the reference medium) are displayed in Fig. 6b. As obtained in the example of **case 2**, the effective mass density can be calculated analytically in this case. For the specific problem in Fig. 6, we have

$$\rho_{\text{eff}} = \langle \rho_1(y_1)^{-1} \rangle^{-1} \langle \rho_1(y_2) \rangle = 1.6 \tag{125}$$

And for the speed of sound,

$$c_{\text{eff}}^2 = \rho_{\text{eff}} \left\langle \frac{\rho_1(y_1)\rho_1(y_2)}{c_1^2(y_1)c_1^2(y_2)} \right\rangle^{-1} = 1.7254 \tag{126}$$

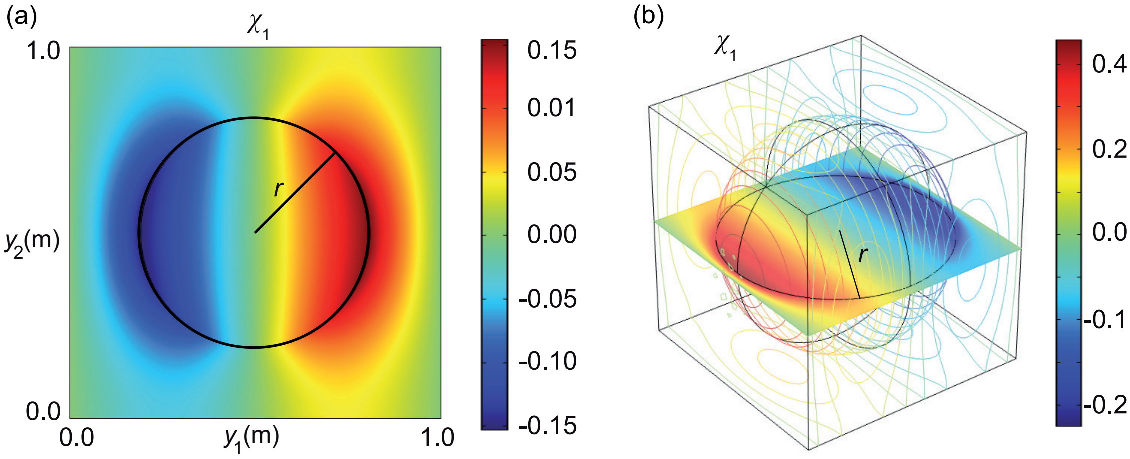

Figure 7: Numerically calculated values of χ_1. (a) 2D case with cylinder radius of 0.3090 m (pressure wave equation) (b) 3D case with sphere radius of 0.4571 m (velocity potential wave equation)

In the case of the pressure wave equation we have

$$\rho_{\text{eff}} = \left(\frac{\langle \rho_1(y_1)^{-1} \rangle}{\langle \rho_1(y_2) \rangle} \right)^{-1} = 1.6 \tag{127}$$

$$c^2_{\text{eff}} = \rho^{-1}_{\text{eff}} \left\langle \frac{1}{\rho_1(y_1)\rho_1(y_2)c^2_1(y_1)c^2_1(y_2)} \right\rangle^{-1} = 1.1391 \tag{128}$$

The effective parameters were also calculated with the numerical method described in the previous section both for the pressure and velocity potential equations, obtaining the same results. As an example, the values of χ_1 in a unit cell are shown in Fig. 6b. Note that the theoretical expressions for the effective density for each case coincide, while this is not the case for the speed of sound.

Once we are confident that the numerical calculation method works, we come back to the homogenization of arrays of cylinders and spheres and calculate the corresponding effective parameters. The values of χ_1 obtained from the simulations are shown in Fig. 7 for two different cases, while the effective parameters are shown in Fig. 8.

In the case of the cylinders, the effective density associated with the homogenized ϕ and p equations is the same. This should be the expected behavior since, in principle, both quantities represent the same physical system (a system described by the same underlying density and sound speed). As a double-check, we also plot the function $(1-f)/(1+f)$ which is known to provide a very good approximation for the effective density in this kind of system for low filling fractions [49]. However, the effective speeds of sound found after the homogenization of each equation are different [see Fig. 8(b)]. We

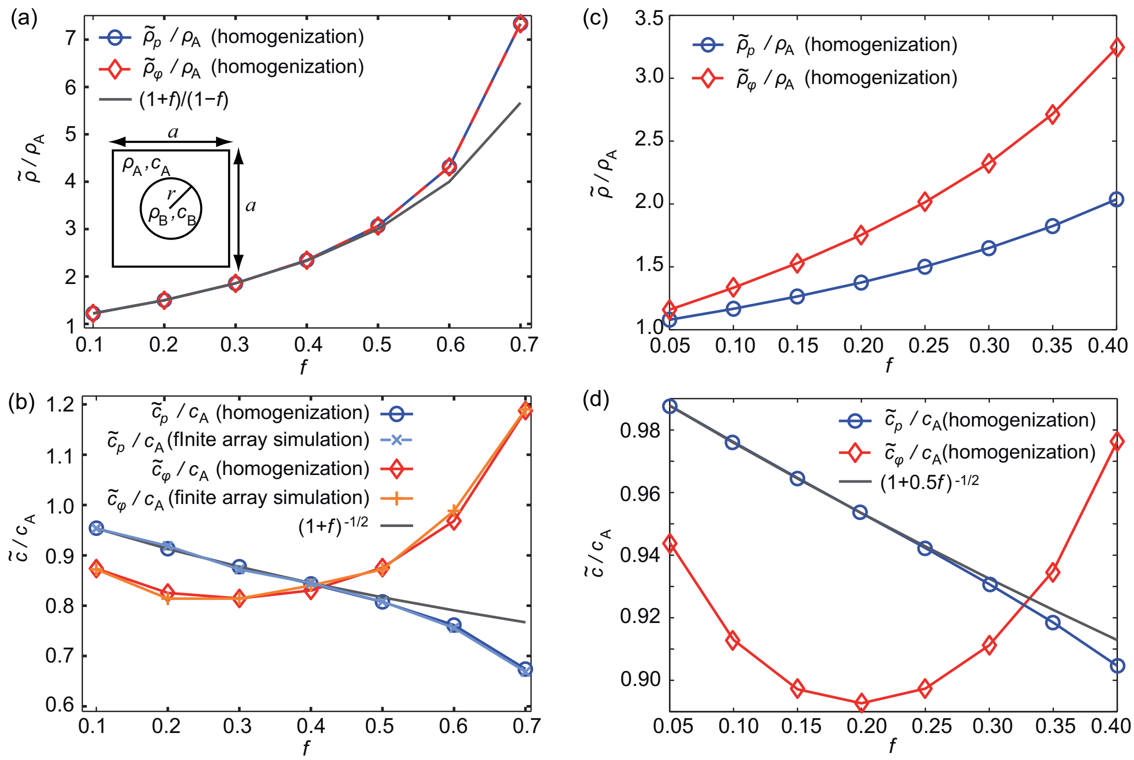

Figure 8: Effective parameters obtained for periodic arrays (square lattice with periodicity l) of wood inclusions embedded in air as a function of the wood filling fraction f ($f = \pi r^2/l^2$ for infinite cylinders and $f = 4\pi r^3/(3l^3)$ for spheres, r being the radius). (a) Effective density and (b) speed of sound of a 2D array of cylinders. The speed of sound is calculated using two different methods (homogenization and simulation of a finite array). (c) Effective density and (d) speed of sound of a 3D array of spheres.

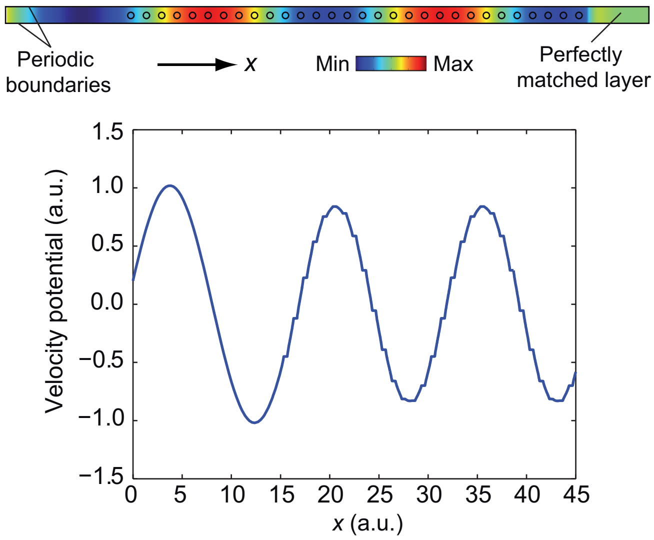

Figure 9: Simulation of the acoustic velocity potential in a finite array of wood cylinders embedded in air. A plane wave impinges onto the array from the left. A perfect matched layer absorbs the output wave on the right. Periodic conditions are applied at the top and bottom simulation boundaries.

further verified these results by simulating a finite array of wood cylinders in COMSOL for each considered 2D configuration. From the simulation we can retrieve the effective speed of sound inside the finite array, which is in excellent agreement with that predicted by the homogenization process. The velocity potential for one of the simulated finite-array configurations is depicted in Fig. 9. In addition, we plot the function $(1-f)^{-1/2}$ in Fig. 8(b), a good approximation for the effective speed of sound at low filling fractions. Remarkably, the values of $\tilde{c}$ arising from this approximation only coincide with those associated with the homogenization of the pressure wave equation.

In the 3D case (wood spheres in air), neither the homogenized densities nor the sound speeds coincide [see Fig. 8(c)-(d)]. Again, only $\tilde{c}_p$ agrees with the prediction for low filling fractions, given in this case by $(1+0.5f)^{-1/2}$. Clearly, these results must imply that a certain distribution of acoustic parameters ρ_0 and c_0 represents a different physical system depending on whether it is used in the ϕ-equation or in the p-equation. In section 2.3.2 we look for an explanation to this behavior.

To conclude this section, we briefly analyze a simple yet useful example that we will employ in the construction of a velocity potential cloak. It is that in which the medium consists of a periodic multilayer structure made up of two different homogeneous materials. This kind of structure falls within the class of systems described by Eq. (113). Specifically, if the two materials are characterized by parameters $\rho_{\rm A}$, $c_{\rm A}$ and $\rho_{\rm B}$, $c_{\rm B}$, and the thickness of the layers of each material is d_{A} and d_{B}, then

$$\tilde{\rho}^x_\phi = \langle \rho(x)^{-1} \rangle^{-1} = \frac{d_{\mathrm{A}} + d_{\mathrm{B}}}{d_{\mathrm{A}} \rho_{\mathrm{A}}^{-1} + d_{\mathrm{B}} \rho_{\mathrm{B}}^{-1}} \tag{129}$$

$$\tilde{\rho}^y_\phi = \tilde{\rho}^z_\phi = \langle \rho(x) \rangle = \frac{d_1 \rho_{\mathrm{A}} + d_{\mathrm{B}} \rho_{\mathrm{B}}}{d_{\mathrm{A}} + d_{\mathrm{B}}} \tag{130}$$

Again, the effective parameters arising from the homogenization of the p-equation for this unidimensional example are different from their homogenized ϕ-equation counterparts. In particular,

$$\tilde{\rho}^x_p = \tilde{\rho}^y_\phi, \tag{131}$$

$$\tilde{\rho}^y_p = \tilde{\rho}^x_\phi. \tag{132}$$

2.3.2 Physical nature of the acoustic medium

Both the p and ϕ equations are derived from the basic principles of fluid mechanics for barotropic fluids, which, in the absence of mass sources and external forces are described by the equations [51]

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \ , \tag{133}$$

$$\rho \mathbf{v} \cdot \nabla \mathbf{v} + \rho \partial_t \mathbf{v} = -\nabla p \ , \tag{134}$$

$$\partial_t s + \mathbf{v} \cdot \nabla s = 0 \ , \tag{135}$$

where s is the entropy. These equations are to be supplemented with the equation of state of the medium, which for the barotropic case can be written as

$$p(x^i, t) = g(\rho, x^i), \tag{136}$$

where $\rho = f(x^i, t)$ is also a function of space and time. Therefore, in order to find an answer to the above conundrum, we looked at the assumptions required to obtain the p and ϕ equations from the previous ones. We will not include here the full derivation of the two equations (the interested reader can consult Refs. [36, 39, 51, 52]), but only the steps required to understand the physical nature of the medium underlying each of them. In both cases, the first step is the linearization of Eqs. (133)–(135). For that, we express all variables as a sum of a background (subscript 0) and an acoustic (subscript 1) contribution, i.e., $\mathbf{v} = \mathbf{v}_0 + \epsilon \mathbf{v}_1$, $p = p_0 + \epsilon p_1$, and $\rho = \rho_0 + \epsilon \rho_1$, with $\epsilon \ll 1$. Substitution of these expressions into Eqs. (133)–(135) gives rise to a set of equations for the background and acoustic variables, from which the p and ϕ equations are derived after imposing some additional conditions. Specifically, to obtain the p-equation one must assume that the background or ambient fluid is isobaric [39], i.e.,

$$\nabla p_0 = 0. \tag{137}$$

On the other hand, to obtain the ϕ-equation, it is necessary to define an enthalpy function h such that [36,51]

$$\nabla h = \frac{\nabla p}{\rho}, \tag{138}$$
$$\frac{\partial h}{\partial p} = \frac{1}{\rho}. \tag{139}$$

These relations are satisfied when h is only a function of p and the equation of state is of the form $p(x^i, t) = g(\rho)$ or, equivalently,

$$\nabla g(\rho_0, x^i) = 0. \tag{140}$$

Then, the enthalpy can be expressed as

$$h(p) = \int_0^p \frac{1}{\rho(p')} dp'. \tag{141}$$

We refer to the class of media described by Eq. (140) as *globally barotropic*, which are a subset of the more general *locally barotropic* media described by Eq. (136).

We can infer more information about the physical systems underlying the p and ϕ equations from the equation of state. In particular, it can be shown that, taking the gradient of Eq. (136) and linearizing, the corresponding equation for the background reads

$$\nabla p_0 = \frac{\partial g(\rho_0, x)}{\partial \rho_0} \nabla \rho_0 + \nabla g(\rho_0, x^i). \tag{142}$$

Thus, the p-equation implicitly requires that

$$\nabla \rho_0 = -\frac{\nabla g(\rho_0, x^i)}{\frac{\partial g(\rho_0, x)}{\partial \rho_0}}, \tag{143}$$

while for the ϕ-equation we have

$$\nabla \rho_0 = \frac{\nabla p_0}{\frac{\partial g(\rho_0, x)}{\partial \rho_0}}. \tag{144}$$

As a result, in the case of the p-equation, inhomogeneities in the acoustic parameters are only allowed if they come from having different media at each point (all of them at the same background pressure). This is the usual configuration employed in the construction of metamaterials [4,16,21–23,26]. However, in the case of the ϕ-equation, we must have the same medium everywhere, although the acoustic parameters may vary from point to point as a consequence of a background pressure gradient. Therefore, although the values of ρ_A and ρ_B, and c_A and c_B in the examples analyzed in section 2.3.1 are the same in both cases, they represent two entirely different physical systems depending on the considered equation. The situation is outlined in Fig. 10. This explains the results of section 2.3.1; the considered acoustic parameters represent the wood-air system only

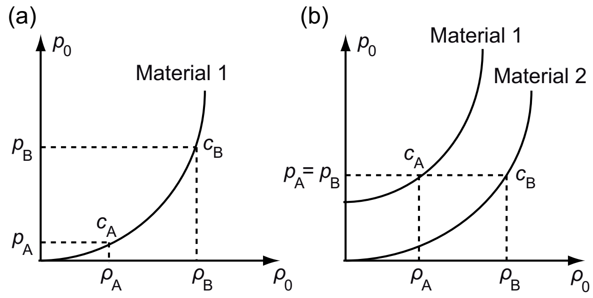

Figure 10: Same set of acoustic parameters arising from two different physical systems. (a) One material at different static pressures (this kind of system is modeled through the ϕ-equation). (b) Two different materials at the same static pressure (this kind of system is modeled through the p-equation).

when plugged into the p-equation, while they represent a fluid subjected to a certain pressure distribution when plugged into the ϕ-equation.

Since only ATA allows us to work with moving background fluids or implement space-time transformations, the above results are crucial for the understanding of which type of physical systems can actually implement ATA's prescriptions. Only the inhomogeneities arising from pressure gradients are permitted by the ϕ-equation around which ATA revolves. Thus, conventional isobaric composites do not qualify for constructing general space-time transformation media with this method. In addition, forcing background pressure gradients in a homogeneous material constitutes an alternative way of building metafluids with properties significantly different from those of isobaric ones. For instance, a remarkable difference that appears in the previous example is the possibility of obtaining supersonic speeds (with respect to the background) for relatively low filling fractions, a feature not displayed by the isobaric composite.

Although the actual attainment of a certain static pressure profile will not be addressed here, we would like to suggest two different potential ways of approaching this issue. The first one is based on the use of one or several pump waves with a much higher amplitude than the acoustic one, which could generate the desired background pressure distribution. The second possibility is related to the transmission of sound in fluids flowing within pipes, where the static pressure depends on the pipe transverse section by Bernoulli's principle. In both scenarios, smooth gradients of the acoustic parameters could be achieved without the need for homogenization. Nonetheless, it is worth pointing out that only the acoustic density can be varied by modifying the background pressure in an ideal gas, since its equation of state is defined by a linear relation. To be able to tailor the speed of sound as well, another more complex material has to be employed, such as the ones governed by polytropic processes.

Finally, we would like to clarify an additional subtlety that we found with respect to the definition of c, where $c = c_0 + \epsilon c_1$. This quantity is commonly taken as [53]

$$c^2 = \frac{\partial p}{\partial \rho} \tag{145}$$

A more general definition is given by the relation [39]

$$(\partial_t + \mathbf{v}\cdot\nabla)p = c^2(\partial_t + \mathbf{v}\cdot\nabla)\rho \tag{146}$$

Both definitions are equivalent for globally barotropic fluids. However, in the locally barotropic case, only the second one remains valid, since it is easy to show that the use of Eq. (145) no longer yields the standard pressure wave equation.

2.3.3 Cloaking the velocity potential

Using the results of section 2.3.1 we can now design devices that implement any spatial transformation of the acoustic velocity potential. For this purpose, we express the ϕ-equation that describes sound propagation in a virtual medium characterized by density $\rho_{\rm V}$ and sound speed $c_{\rm V}$ in arbitrary spatial coordinates x^i

$$-\frac{\rho_V}{c_{\rm V}^2}\partial_t^2\phi_{1\rm V} + \frac{1}{\sqrt{\gamma}}\partial_i\left(\rho_{\rm V}\sqrt{\gamma}\gamma^{ij}\partial_j\phi_{1\rm V}\right) = 0. \tag{147}$$

Under a purely spatial coordinate transformation $\bar{x}^i = f(x^i)$, this equation becomes

$$-\frac{\rho_V}{c_{\rm V}^2}\partial_t^2\bar{\phi}_{1\rm V} + \frac{1}{\sqrt{\bar{\gamma}}}\partial_i\left(\rho_{\rm V}\sqrt{\bar{\gamma}}\bar{\gamma}^{ij}\partial_j\bar{\phi}_{1\rm V}\right) = 0, \tag{148}$$

where $\bar{\gamma}^{\bar{i}\bar{j}} = \Lambda_i^{\bar{i}}\Lambda_j^{\bar{j}}\gamma^{ij}$, $\Lambda_i^{\bar{i}}$ being the Jacobian of the transformation. In order to mimic the distortion introduced by this transformation we consider a second (real) medium consisting of a microstructure characterized by the position-dependent parameters $\rho_{\rm R}$ and $c_{\rm R}$. As deduced above, sound waves satisfy Eq. (104) in the long-wavelength regime. This equation can also be expressed in coordinates x^i yielding

$$-\langle\frac{\rho_{\rm R}}{c_{\rm R}^2}\rangle\partial_t^2\phi_{1\rm R} + \frac{1}{\sqrt{\gamma}}\partial_i\left(\sqrt{\gamma}\tilde{\rho}_{\rm R}^{ij}\partial_j\phi_{1\rm R}\right) = 0, \tag{149}$$

where the coefficients $\tilde{\rho}_{\rm R}^{ij}$ are obtained via Eq. (118). Note that this equation is still valid when $\tilde{\rho}_{\rm R}^{ij}$ has a slow spatial variation (slow spatial variation of the microstructure properties). Clearly, if we rename $\bar{x}^i$ to x^i in Eq. (148), then Eqs. (148) and (149) are formally identical if

$$\frac{\tilde{\rho}_{\rm R}^{ij}}{\rho_{\rm V}} = \frac{\sqrt{\bar{\gamma}}}{\sqrt{\gamma}}\bar{\gamma}^{ij}; \tag{150}$$

$$\langle\frac{\rho_{\rm R}}{c_{\rm R}^2}\rangle = \frac{\sqrt{\bar{\gamma}}}{\sqrt{\gamma}}\frac{\rho_V}{c_{\rm V}^2}. \tag{151}$$

Example: cylindrical cloak

Consider the following radial transformation in cylindrical coordinates [13, 23]:

$$\begin{aligned}\bar{r} &= \frac{b-a}{b}r + a \ ; \\ \bar{\theta} &= \theta \ ; \\ \bar{z} &= z \ . \end{aligned} \tag{152}$$

According to the previous results, the effective parameters required for this transformation are

$$\rho^{ij} = \rho_{\rm V} \begin{pmatrix} \frac{r-a}{r} & 0 & 0 \\ 0 & \frac{r}{r-a} & 0 \\ 0 & 0 & \frac{r-a}{r}\left(\frac{b}{b-a}\right)^2 \end{pmatrix} ; \tag{153}$$

$$\langle \frac{\rho_{\rm R}}{c_{\rm R}^2} \rangle = \frac{r-a}{r}\left(\frac{b}{b-a}\right)^2 \frac{\rho_V}{c_{\rm V}^2}. \tag{154}$$

To implement these parameters, we consider a cylindrical multilayer structure made up of alternating low- and high-density materials characterized by parameters $\rho_{\rm A}$, $c_{\rm A}$ and $\rho_{\rm B}$, $c_{\rm B}$, respectively. Again, the length of each low- (high-) density layer is supposed to be $d_{\rm A}$ ($d_{\rm B}$). To a good approximation, we can use the results of Eqs. (129) and (130) to obtain the effective density of this structure by taking $\tilde{\rho}^r_\phi = \tilde{\rho}^x_\phi$ and $\tilde{\rho}^\theta_\phi = \tilde{\rho}^y_\phi$. Following the procedure of Ref. [23], we simplify the design by setting the thickness of all layers to a fixed value $d_{\rm A} = d_{\rm B} = d$ and impose the following conditions

$$\frac{\rho_{\rm A}}{\rho_{\rm V}} = \frac{\rho_{\rm V}}{\rho_{\rm B}}; \tag{155}$$

$$c_{\rm A} = c_{\rm B}. \tag{156}$$

As a result, we obtain

$$\tilde{\rho}^r_\phi = \frac{2}{\rho_{\rm A}^{-1} + \rho_{\rm B}^{-1}}; \tag{157}$$

$$\tilde{\rho}^\theta_\phi = \langle \rho(r) \rangle = \frac{\rho_{\rm A} + \rho_{\rm B}}{2}; \tag{158}$$

$$\langle \frac{\rho_{\rm R}}{c_{\rm R}^2} \rangle = \frac{\rho_{\rm A} + \rho_{\rm B}}{2c_{\rm A}^2}. \tag{159}$$

For these values to be equal to those specified by Eqs. (153)–(154), it is clear that

$$\rho_{\rm A} = \frac{\rho_{\rm V}}{r-a}\left(r + \sqrt{2ar - a^2}\right); \tag{160}$$

$$c_{\rm A} = \frac{r}{r-a}\frac{b-a}{b}c_{\rm V}. \tag{161}$$

Figure 11: Gaussian beam (a) propagating through air, (b) impinging onto a high-density cylinder, (c) and onto the cloak-surrounded cylinder. The cloak consists of 50 different layers with $b = 2a = 1.7\lambda$, and $d \approx \lambda/100$, where λ is the acoustic wavelength.

To verify the functionality of the designed cloak we solved the velocity potential equation in COMSOL for different situations (see Fig. 11). First, we simulated a Gaussian beam propagating through air. If a high-density cylinder is placed in its way, there appear shadows and reflections. These effects are suppressed if the cylinder is surrounded by the designed multilayer. According to our previous analysis, note that the whole cloak should consist of a unique fluid, while the different densities and sound speeds required at each region (layer) should be obtained by forcing the corresponding background pressure.

In addition, note that this cloak implements the transformation given by Eq. (152) over the velocity potential. Therefore, the pressure distribution inside the cloak is different from the one existing in a device designed to cloak the pressure, as the one in [23]. Nevertheless, both devices prevent the acoustic wave from entering the inner region, which is equally cloaked in the two cases.

3 Transformation Elasticity

Given the success of transformational techniques in optics and now in acoustics, it is natural to ask whether similar ideas can be exported to other fields of physics. In this second part of the report we evaluate this question for the important particular case of elasticity theory. The ultimate objective would be to achieve a transformational framework for elastic waves, i.e. to be able to apply techniques of transformational physics to the general theory of linear elasticity. For example, a generic linear elastic tensor has 21 components, and the different possible elastic waves propagate and interact in a complicated way. However, in many common cases this number is strongly reduced. In the well-known case of Lamé elastic systems, for instance, which are homogeneous and isotropic, the number of independent components is reduced to only two parameters, the first Lamé parameter λ and the shear modulus or second Lamé parameter μ. The corresponding elastic waves in such Lamé systems neatly separate into longitudinal and transversal elastic wave modes which evolve independently. This has important applications, amongst others, in seismology, where the two wave types are called P (primary) and S (secondary) waves, see e.g. [54]. With a hypothetical transformation elasticity, complicated media could inherit much simpler properties, for example similar to Lamé systems.

In this report we discuss the problems encountered when trying to build a complete transformational framework for elasticity. We propose different approaches one could take and discuss how far one could go in each one of them.

3.1 Standard elasticity equations and their transformation properties

The first question to ask in the context of transformation elasticity is: what are the transformation properties of the elastic wave equation? In Cartesian coordinates, the standard textbook equation for elastic waves [55] is

$$\partial_i C^{ijkl} \partial_j u_k - \rho \partial_0^2 u_l = 0 \, , \tag{162}$$

with C^{ijkl} the elastic tensor, u_k the displacement vector and ρ the density, and as usual latin indices $(1-3)$ indicate spatial dimensions while 0 is the time index. When asked about the transformation properties of this equation, this is no longer a textbook application.

A first guess, based on a straightforward "tensorialization" of the previous equation, suggests that it would read

$$\nabla(\mathbf{C}\colon \nabla\mathbf{u}) = \rho\ddot{\mathbf{u}} \ , \tag{163}$$

with the so-called double-dot product defined as $\mathbf{X}\colon \mathbf{Y} = X^{i_1 i_2 \ldots i_{m-1} i_m} Y_{i_{m-1} i_m j_3 \ldots j_n}$ (i.e., the last 2 indices of $\mathbf{X}$ are contracted with the first 2 of $\mathbf{Y}$).

This equation is tensorial under general spatial changes of coordinates, and can readily be used in transformation elasticity in a very powerful way (although restricted to spatial transformations, since one cannot mix space and time). However, we need to make sure that this equation is indeed correct, at least for "basic" elasticity (i.e., without necessarily thinking of composite materials and more intricate structures), or whether it becomes more complicated when written in arbitrary coordinates. This is the subject of the present section.

Let us look at the textbook equations for linear elasticity and see how they can be written tensorially. For the moment, we think of simple (non-composite) materials, and are only interested in the transformation properties of the equations.

3.1.1 Constitutive equation

The constitutive law for linear elasticity is the generalized Hooke's law, which can be written in (spatial) tensorial form as

$$\boldsymbol{\sigma} = \mathbf{C}\colon \boldsymbol{\varepsilon}, \tag{164}$$

or, in component notation:

$$\sigma^{ij} = C^{ijkl}\varepsilon_{kl}, \tag{165}$$

and expresses the relation between the (Cauchy) stress tensor $\boldsymbol{\sigma}$ and the strain tensor $\boldsymbol{\varepsilon}$. Note that, in linear elasticity, both $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are symmetric:

$$\sigma_{ij} = \sigma_{ji}, \qquad \varepsilon_{ij} = \varepsilon_{ji}. \tag{166}$$

The elastic or stiffness tensor $\mathbf{C}$ a priori has $3^4 = 81$ entries, but these are reduced because of the symmetry relations just expressed:

- $\sigma_{ij} = \sigma_{ji}$ implies $C^{ijkl} = C^{jikl}$
- $\varepsilon_{ij} = \varepsilon_{ji}$ implies $C^{ijkl} = C^{ijlk}$

leaving 36 independent elastic coefficients. Moreover, one usually assumes hyperelasticity, and

- hyperelasticity implies $C^{ijkl} = C^{klij}$, so only 21 independent components remain. The condition for hyperelasticity is that the constitutive equation can be derived from a strain energy density function $\mathcal{U}(E, x)$. In the linear regime, this means

$$\sigma^{ij} = \frac{\partial \mathcal{U}}{\partial \epsilon_{ij}}, \tag{167}$$

which, in the case of linear elasticity, can be shown to be equivalent to the symmetry condition on the tensor $\mathbf{C}$ expressed above.[2] One also has

$$\mathcal{U} = \frac{1}{2}\sigma^{ij}\epsilon_{ij}. \tag{169}$$

Any symmetry properties of the material further reduce the number of independent components of $\mathbf{C}$. For example, for isotropic materials, the elastic moduli can be expressed in terms of only 2 independent parameters, Lamé's (first) parameter λ and the shear modulus or Lamé's second parameter μ, such that

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{170}$$

which can be generalized to

$$C_{ijkl} = \lambda g_{ij}g_{kl} + \mu(g_{ik}g_{jl} + g_{il}g_{jk}) \tag{171}$$

with g_{ij} the spatial metric.

Note that, for elasticity to be in the linear regime, the displacement gradients must be infinitesimal. This does not imply that the displacements themselves should be infinitesimal. Under this condition of infinitesimal displacement gradients, the Cauchy stress tensor $\boldsymbol{\sigma}$ coincides with the so-called first and second Piola-Kirchhoff tensors, which are also often encountered in the literature, see e.g. [56].

[2] Note that hyperelasticity is a different assumption from linear elasticity, but the former is a more general assumption than the latter: linear elasticity models can be described as a form of hyperelasticity, by solving a set of compatibility conditions, but the reverse is not always true. In fact the hypothesis of hyperelasticity is typically introduced to extend the analytical treatment of linear elasticity to non-linear cases. [56]. Note, however, that there exist energy stability conditions in hyperelastic materials which impose certain limits on the allowable elastic parameters. The elastic energy $\mathcal{U}$ has to be positive definite for all non-zero strains in order to guarantee stability under elastic fluctuations. For the important class of isotropic crystals, this energy stability condition reduces to

$$\mu > 0; \qquad \kappa \equiv \lambda + \frac{2}{3}\mu > 0, \tag{168}$$

with λ and μ the Lamé parameters for isotropic materials and where the so-called bulk modulus κ is a measure for the (in)compressibility of the material. However, these conditions do not necessarily hold for composite materials, and indeed one of the key indicators of an upcoming breakthrough in transformation elasticity is the existence in the literature of (proposed) elastic metamaterials with effective values $\rho_{\text{eff}} < 0$, $\kappa_{\text{eff}} < 0$ and/or $\mu_{\text{eff}} < 0$, see e.g. [57–59] (or even complex-valued effective parameters, see the example mentioned in Sec. 3.3).

3.1.2 Strain-displacement relation

The relation between the strain tensor $\boldsymbol{\varepsilon}$ and the displacement vector $\mathbf{u}$ is given by

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right]. \tag{172}$$

In a Cartesian coordinate system, this can be written in component notation as

$$\varepsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) = \partial_{(i}u_{j)}. \tag{173}$$

One can also define the rotation tensor $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} = \frac{1}{2}\left[\nabla\mathbf{u} - (\nabla\mathbf{u})^T\right], \tag{174}$$

which, in a Cartesian coordinate system, equals

$$\omega_{ij} = \frac{1}{2}\left(\partial_i u_j - \partial_j u_i\right) = \partial_{[i}u_{j]}. \tag{175}$$

Note that $\nabla\mathbf{u} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}$, whereas $\nabla\cdot\mathbf{u} = \mathrm{tr}\,\boldsymbol{\varepsilon}$.

3.1.3 Equation of motion

The equation of motion for the displacement vector $\mathbf{u}$ is

$$\mathbf{f}_{\mathrm{ext}} + \nabla\cdot\boldsymbol{\sigma} = \rho\ddot{\mathbf{u}}, \tag{176}$$

with $\mathbf{f}_{\mathrm{ext}}$ the external force.

3.1.4 Elastic wave equation

Note the tensorial character of all three previous equations: the constitutive equation, the strain-displacement relation, and the equation of motion, as stressed e.g. in [56]. Together these form a set of 15 (6+6+3) scalar field equations in any coordinate system. One can combine these three equations to obtain a wave equation in terms of the displacement vector $\mathbf{u}$. Inserting $\boldsymbol{\varepsilon}$ from Eq. (172) into Eq. (164) leads to

$$\boldsymbol{\sigma} = \frac{1}{2}\mathbf{C}\colon\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right]. \tag{177}$$

Plugging this into Eq. (176) gives

$$\frac{1}{2}\nabla\left\{\mathbf{C}\colon\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right]\right\} + \mathbf{f}_{\mathrm{ext}} = \rho\ddot{\mathbf{u}}. \tag{178}$$

Because of the symmetry properties of the elasticity tensor $\mathbf{C}$, this can be simplified[3] to

$$\nabla\cdot(\mathbf{C}\colon\nabla\mathbf{u}) + \mathbf{f}_{\mathrm{ext}} = \rho\ddot{\mathbf{u}}. \tag{179}$$

[3]This is easy to see in component notation: $\mathbf{C}\colon\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right]$ becomes $C^{ijkl}(\partial_k u_l + \partial_l u_k)$. Since $C^{ijkl} = C^{ijlk}$, one immediately obtains the desired result.

This is the general wave equation in the linear elasticity regime for the displacement vector $\mathbf{u}$ in tensor form.[4] In Cartesian coordinates, this becomes

$$\partial_i \left(C^{ijkl} \partial_k u_l \right) + f^j_{\text{ext}} = \rho \partial_t^2 u^j. \tag{183}$$

The well-known Navier-Lamé equations

$$(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} + \mathbf{f} = 0 \tag{184}$$

can be obtained from Eq. (179) by assuming isotropy [Eq. (170)] and equilibrium ($\ddot{\mathbf{u}} = 0$).

In conclusion of this section: analyzing the different equations involved in linear elasticity, one convinces oneself that each one of them can be understood as spatial tensorial equations. The crucial point is the identification of the very transformational nature of the objects used in formulating the theory: one is only using 3-dimensional vectors and tensors under spatial changes of coordinates. Once Eq. (162) is written in spatial tensorial form, Eq. (179), its transformation properties are explicit.

Now, given this transformational character, one realizes that one can make full use of generic spatial coordinate transformations to generate elastic devices. For instance one could engineer a very complicated C^{ijkl} that nonetheless behaves as a simple homogeneous Landé system with its decoupling of transverse and longitudinal waves.

The next step is then to look at whether one could also apply spacetime transformations.

3.2 Spacetime transformations: an abstract relativistic system

The physical equation of standard linear elasticity are no longer form-invariant when applying general transformations that mix space and time. The situation is similar to the one with the acoustic equation in fluids. Thus, in principle there is an obstruction to the direct application of transformational techniques. However, we have previously shown that this problem can be overcome in acoustics by using an intermediate abstract system with the "desirable" space-time transformation properties [20, 31]. In the case of

[4] Curiously, it is hard to find this equation written explicitly in the literature. Most textbooks focus immediately on the isotropic case and/or on Cartesian coordinates, and derive the Navier-Lamé equations without going through the more general wave equation Eq. (179) first. E.g., Ref. [17] gives

$$\nabla \cdot \boldsymbol{\sigma} = -\omega^2 \rho \mathbf{u}, \qquad \boldsymbol{\sigma} = \mathbf{C} \nabla \mathbf{u}, \tag{180}$$

but fails to combine them into a single wave equation for $\mathbf{u}$. Similarly, [18] writes

$$\text{div} \boldsymbol{\sigma} = \rho^{\text{eff}} \ddot{\mathbf{u}}, \qquad \boldsymbol{\sigma} = \mathbf{C}^{\text{eff}} \nabla \mathbf{u}, \tag{181}$$

again without combining them into a single wave equation. The motivation in both cases to keep the two equations separate seems to radicate in the possibility of having more complicated (e.g., nonsymmetric) stress tensors $\boldsymbol{\sigma}$.

As a rare exception, see the online course [60] which gives the Fourier-transformed version of Eq. (179) (in the absence of external forces):

$$\nabla \cdot (\mathbf{C} \colon \nabla \mathbf{u}) + \omega^2 \rho \mathbf{u} = 0. \tag{182}$$

acoustics this system was a relativistic Klein-Gordon field living in a curved spacetime manifold. Here we want to evaluate whether a method similar to the one we applied in acoustics could work for elasticity also.

One way to proceed is the following. Recall that the crucial step in the construction of the Klein-Gordon equation (45) for the (scalar) velocity potential perturbation ϕ_1 in acoustics was to write the acoustic equation (3) in the form

$$\partial_\mu f^{\mu\nu} \partial_\nu \phi_1 = 0, \tag{185}$$

and then identifying $f^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$. In the case of elasticity, the central wave equation (179) concerns a vector, the displacement vector $\mathbf{u}$, and so any relativistic counterpart will necessarily be more complicated.

However, in analogy with Eq. (185), one could blindly write a system of 4-dimensional relativistic equations as

$$\nabla_\mu f^{\mu\nu\sigma\tau} \nabla_\nu \bar{u}_\sigma = 0, \tag{186}$$

with $\bar{\mathbf{u}}$ the adequate four-vector (spacetime) generalization of the displacement three-vector $\mathbf{u}$.

By construction we assume that this equation is spacetime tensorial and nothing depends on coordinates. At this stage one may consider that the four-dimensional manifold is equipped with a simple flat metric so that the nabla derivatives (∇_μ, ∇_ν) represent the standard flat covariant derivatives. It is not necessary to be able to identify this equation with some common relativistic theory; to our purposes it is only a possible relativistic mathematical model.[5]

Then, one can check that, using Cartesian coordinates and for a particular tensor $f^{\mu\nu\sigma\tau}$, this equation is formally equal to the elasticity equation

$$\partial_i C^{ijkl} \partial_j u_k - \rho \partial_0^2 u_l = 0. \tag{187}$$

For instance, as a first condition, take $u^0 = 0$. Then define

$$f^{ijkl} = C^{ijkl}; \tag{188}$$

$$f^{0j\sigma\tau} = f^{i0\sigma\tau} = 0; \tag{189}$$

$$f^{00kl} = -\rho\delta^{kl}. \tag{190}$$

Also

$$f^{00i0} = f^{000j} = 0; \tag{191}$$

$$f^{0000} = 0. \tag{192}$$

One could also reduce the last 3 equations by writing $f^{00\sigma\tau} = -\rho\delta^{\sigma\tau}$. Even though this would make $f^{0000} = -\rho$, seemingly in contradiction with Eq. (192), this is compensated by the fact that $u_0 = 0$ at this point.

[5] It is nonetheless interesting to realize that this equation appears in the Carter-Quintana formulation of relativistic elasticity theory [61]. In this formulation all objects are by construction spacetime tensors, and $f^{\mu\nu\sigma\lambda}$ has to satisfy some orthogonality requirements so that it can be associated to a medium existing in space.

Now one can change coordinates in the specific abstract system defined above and see what the transformation leads to. The central question is: are the new tensors also identifiable with a (different) set of material parameters for some familiar type of elasticity? At first sight, there are two elements that complicate the answer. First of all, a generic spacetime-mixing change of coordinates would make the new u^0 different from zero. How should one interpret this apparently new degree of freedom in the resulting equation? To avoid adding new degrees of freedom into the theory, we can take the four-vector in the relativistic abstract system to be a normal unit spacelike vector $u^\mu u_\mu = 1$. Then, one can always eliminate u^0 in the equations as a function of the u^i. The other apparent problem is the existence of a fourth equation of motion for the u^i, the $\tau = 0$ component in Eq. (186). However, this additional equation does not constrain the elastic system further as one has the freedom to assume any convenient $f^{\mu\nu\sigma 0}$, which do not appear in the other three equations. Thus, one can simply ignore this fourth equation. Therefore, one obtains some sort of elastic theory (three equations for the three u^i) which has mixed terms $\partial_t \partial_i$ and some of its terms contain $u^0 = f(u^i)$.

In one respect the situation is not surprising. We have learned from acoustics that spacetime transformations, and in particular mixed terms $\partial_t \partial_i$, will typically require background flows [20]. However, elastic waves are generally associated with solids, where such background flows are less obvious to realize.[6] Similar to the method suggested in Sec. 2.3.2 to obtain inhomogeneous acoustic parameters in the velocity-potential formalism, here also one might resort to non-linear situations in which the presence of a moving background wave acts as a changing background for smaller linear perturbations on top of it.

The other unfamiliar ingredient in the resulting family of equations is the appearance of terms with non-linear functions of u^i. In general situations these non-linear functions will contain $u^i u_i$ and $g_{0i} u^i$ terms. This last term could be associated to a form of background flow $v_i = g_{0i}$. At the end of the day one should be able to engineer a system of non-linear elasticity containing terms depending on the norm of the displacement vector and on its projection on the background flow. This is one of the lines of future investigation we are pursuing.

3.3 A physically different elastic equation

Let us come back to purely spatial changes of coordinates. In this restricted setting one could explore the possibility of connecting the elastic equation (162) with an (in principle abstract) system different from the physical one described by (163). Imagine for instance that one rewrites the (by definition: contravariant) displacement vector u^i as $\delta^{ij} u_i$ and ascribes to δ^{ij} a collection of constant scalars and to u_i a covariant vector. Then, the new abstract equation does not maintain its form even under spatial changes of coordinates. However, what was shown by Milton et al. [17] is that the transformed equation can be realized in certain elastic composite structures and random media. The

[6] Viscoelastic materials might be an interesting exception. But the theory of viscoelasticity is much more complicated than standard linear elasticity and far beyond the scope of this or any other current study on transformation elasticity.

new equations have precisely the form found previously by Willis for this type of systems (see e.g. [32, 33] and references therein):

$$\nabla \boldsymbol{\sigma} = \dot{\mathbf{p}} \tag{193}$$

$$\boldsymbol{\sigma} = \mathbf{C}_{\text{eff}} * \boldsymbol{\varepsilon} + \mathbf{S}_{\text{eff}} * \dot{\mathbf{u}} \tag{194}$$

$$\mathbf{p} = \mathbf{S}^{\dagger}_{\text{eff}} * \boldsymbol{\varepsilon} + \boldsymbol{\rho}_{\text{eff}} * \dot{\mathbf{u}}, \tag{195}$$

where $*$ represents convolution in time. This kind of analysis opens up the possibility of building cloaking devices precisely with such composites and random media. Let us describe briefly the main characteristics of the Willis equations.

3.3.1 Extended elasticity: Willis equations

There exist situations in which the standard linear elasticity equations are not appropriate. For instance this happens in random or composite materials. These systems satisfy the so-called Willis equations [32, 33].

The essential idea behind the Willis equations can be understood as follows. One introduces a (simple and homogeneous) reference material with elastic tensor and density $\mathbf{C}_0, \rho_0$ such that

$$\boldsymbol{\sigma} = \mathbf{C}_0 * \boldsymbol{\varepsilon}, \qquad \mathbf{p} = \rho_0 \dot{\mathbf{u}}. \tag{196}$$

which would give a reference solution $\mathbf{u}_0$ under the conditions of interest (external force $\mathbf{f}_{\text{ext}}$, strain $\boldsymbol{\varepsilon}$). The solution for the random material $\mathbf{C}, \rho$ can then formally be written as the corresponding "inhomogeneous" solution in terms of Green functions $\mathbf{G}$ as

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{G} * (\nabla \cdot \boldsymbol{\tau} - \dot{\boldsymbol{\pi}}), \tag{197}$$

where $\boldsymbol{\tau}, \boldsymbol{\pi}$ encode the difference between the random and the reference material:

$$\boldsymbol{\tau} \equiv (\mathbf{C} - \mathbf{C}_0) * \boldsymbol{\varepsilon}, \qquad \boldsymbol{\pi} \equiv (\rho - \rho_0) \dot{\mathbf{u}}. \tag{198}$$

Taking spatial and time derivatives of (197) then leads to the following pair of equations

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 - \mathbf{S}_x * \boldsymbol{\tau} - \mathbf{M}_x * \boldsymbol{\pi}, \tag{199}$$

$$\dot{\mathbf{u}} = \dot{\mathbf{u}}_0 - \mathbf{S}_t * \boldsymbol{\tau} - \mathbf{M}_t * \boldsymbol{\pi}, \tag{200}$$

where the strain-displacement relation (172) was used in the first to write $\boldsymbol{\varepsilon}$ in terms of $\nabla \mathbf{u}$, and $\mathbf{S}_x$, $\mathbf{S}_t$, $\mathbf{M}_x$ and $\mathbf{M}_t$ are the adequately defined spatial and time derivatives of $\mathbf{G} * \nabla \cdot \boldsymbol{\tau}$ and $\mathbf{G} * \dot{\boldsymbol{\pi}}$, respectively.[7]

Eqs. (199)–(200) can then be plugged into (196), using the definitions (198) of $\boldsymbol{\tau}$ and $\boldsymbol{\pi}$, to find an expression for $\boldsymbol{\sigma}$ and $\mathbf{p}$:

$$\boldsymbol{\sigma} = \mathbf{C}_{\text{eff}} * \boldsymbol{\varepsilon} + \mathbf{S}_{\text{eff}} * \dot{\mathbf{u}}, \tag{201}$$

$$\mathbf{p} = \mathbf{S}^{\dagger}_{\text{eff}} * \boldsymbol{\varepsilon} + \boldsymbol{\rho}_{\text{eff}} * \dot{\mathbf{u}}, \tag{202}$$

[7] A detailed analysis shows that, in fact, $\mathbf{S}_t = \mathbf{M}_x^{\dagger}$, which explains the connection through the conjugate $\mathbf{S}_{\text{eff}}$ and $\mathbf{S}^{\dagger}_{\text{eff}}$ between the two Willis equations (201) below.

where $\mathbf{S}_{\text{eff}}$, but also $\mathbf{C}_{\text{eff}}$ and $\boldsymbol{\rho}_{\text{eff}}$, contain complicated dependencies on the matrices $\mathbf{S}$ and $\mathbf{M}$.

These are the Willis equations, where $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\mathbf{p}$ and $\dot{\mathbf{u}}$ should be interpreted as ensemble-averaged values, i.e. $\langle\boldsymbol{\sigma}\rangle$, $\langle\boldsymbol{\varepsilon}\rangle$, etc.

The key feature of the Willis equations is their generality, and the fact that they are in principle closed under transformations, i.e.: starting from a material which obeys the Willis equations, transforming it will modify the parameters $\mathbf{C}_{\text{eff}}$, $\mathbf{S}_{\text{eff}}$ and $\boldsymbol{\rho}_{\text{eff}}$, but not the form of the equations.

However, an important and immediately apparent drawback is the formal character of the construction. Ultimately, this means that exact expressions for $\mathbf{C}_{\text{eff}}$, $\mathbf{S}_{\text{eff}}$ and $\boldsymbol{\rho}_{\text{eff}}$ cannot be found (not even in principle) for any but the most trivial cases, and one needs to resort to complicated approximation schemes. The connection with real-life (meta)material parameters is so far, in the best of cases, highly non-trivial, see e.g. [62].

A final remark is in order with respect to both the validity and the necessity of working with the Willis equations when dealing with composite materials. While the Willis equations were derived for random media, the validity is in principle restricted to small departures from a homogeneous reference material. On the other hand, for periodically structured composite materials in the limit where the wavelengths (and attenuation lengths) of the waves of interest are much longer than the characteristic distances over which the material parameters vary, there is no need to apply the Willis equations and one can stick to the usual equations of elasticity in terms of averaged values:

$$\nabla \cdot (\mathbf{C}_{\text{eff}} \colon \nabla \mathbf{u}_0) = \langle \rho_\eta \rangle \ddot{\mathbf{u}}_0. \tag{203}$$

The calculation of the effective material parameters in this so-called quasistatic limit is a textbook exercise (see e.g. [60] or [63]). Complications may arise, e.g. the effective tensors may become complex-valued.[8] But it is important to bear in mind that current-day metamaterials for acoustic applications fall precisely in the category where the quasistatic limit would be valid: they consist of periodically arranged arrays of (for example) cylinders in a homogeneous background, with a distance between the cylinders much smaller than the acoustic wavelengths one tries to manipulate. For similar arrangements with elastic metamaterials, one could then use the effective equations (203) without any need to invoke the Willis equations.

3.4 Another tensorial transformation capable of mixing space and time

Milton at al.'s proposal [17] makes us reflect as to whether one can use even more general, although perhaps counterintuitive, transformation laws. One could assume

[8] Ref. [60] gives the example of gas bubbles in water, where the effective composite bulk modulus κ_{eff} acquires an imaginary part determined by the shear modulus of water (and hence sound is strongly damped in the resulting bubbly fluid), whereas κ is always real and positive in "normal" materials (see the remark on hyperelasticity in section 3.1 above). Note that the complex character of the effective bulk modulus here is a direct consequence of the viscous character of water, i.e.: the shear modulus of water is taken to be complex, and this leads to a complex value for the effective bulk modulus of the composite (water+gas).

that the intermediate abstract system is not purely covariant in the general relativity sense. Instead of writing Eq. (163), we could take the u^i in Eq. (162) to be a spatial vector and a temporal scalar. Take the rest of objects in the equation to transform as spatiotemporal tensors [as in Eq. (163)]. On the one hand, this transformation law spoils the form-invariance of the equation under general changes of coordinates. On the other hand, when transformed, no u^0 component appears so that the resulting theory is also linear in the displacement u^i. Again, the crucial thing to check is whether one can reproduce the resulting equations with the use of metamaterial design. For this scheme to make sense, it would be sufficient if one could do that at least for some specific type of constitutive relations.

3.5 More general transformational properties

After the work of Milton et al., Norris and Shuvalov [18] went further and proposed that the matrix A used to transform u^i, i.e.: a matrix $A^j{}_i$ such that $u'^j = A^j{}_i u^i$, can be chosen arbitrarily and independently of the matrix M associated with the change of coordinates, $M^j{}_i = \partial x'^j / \partial x^i$. With a generic transformation the object u^i might fail to be a tensor of any kind. They did not explicitly discuss why this would be allowed, though.

From our perspective based on the construction of an abstract intermediate system, the Norris-Shuvalov proposal can be interpreted as imagining a non-tensorial abstract system with precisely that transformation properties. The crucial point is whether under transformations one is taken to a system of equations one is familiar with, or at least there are realistic prospects of realizing it in a laboratory. It seems that [18] proposes $A^j{}_i \neq M^j{}_i$ because this would allow to transfer the standard elasticity equations into elastic equations for composites, which enter directly into the realm of metamaterials.

It might seem that this discussion implies that finding a form-invariant set of equations is not necessarily the crucial point after all, then. This depends, however, on the perpective taken. Imagine that one starts from the standard elastic equation and one transforms it using some particular transformation law, not necessarily tensorial, as in the Norris-Shuvalov proposal. In general, the final outcome results in a family of equations apparently unrelated to the initial equation. Now, one can always take this new family as the very definition of a "first-order-extended" elasticity theory. One has to transform this first-order-extended elasticity and see whether its form is already closed under the transformation law, such that further transforming it would lead to a structure of the same "extended-elastic family". If not, one would extend further the initial family by constructing a second-order-extended theory, a third-order-extended one, and so on. In this way one would eventually end up with a final extended elasticity with form-invariant equations, in which any transformation of the kind contemplated would lead to a material structure of the same n-extended-elasticity family. So, at the end of the day one really should look at form-invariant equations, but having in mind that maybe the family one starts with does not cover all the systems connected through the transformations.

4 Conclusions and Perspectives

In this report we have deepened our understanding on ATA and studied the extension of the analogue transformations concept to the field of elasticity. We have thus divided this section into two parts corresponding to these main blocks.

4.1 Analogue transformation acoustics

In this first part of the work, we have clarified several fundamental differences between this technique and standard transformation acoustics (STA) building on the results of [31]. First, we have derived the set of transformations that do not preserve the form of the velocity potential equation in the original laboratory space. For these transformations, it is indispensable to apply the ATA method by constructing an auxiliary relativistic analogue spacetime before mapping back to the laboratory spacetime. As a second result, we have shown that the pressure wave equation commonly used in STA is not suitable for building an analogue transformational method, and have highlighted the importance of the background velocity. Third, we have examined in detail three spacetime transformations that can only be performed with ATA. The first one allowed us to design an acoustic frequency converter, the second one was used to engineer a device that is able to increase the density of events within a given region by simultaneously compressing space and time, while the third one allowed to dynamically cloak a certain set of spacetime events during a limited time interval.

Overall, these results confirm the conclusion given in [31]: the different requirements that ATA imposes on acoustic metamaterial design compared to the standard approaches, and even more so: the potential of ATA to design applications which could so far not be obtained with any other methodology, open a completely new perspective on the field of Transformation Acoustics.

As a fourth contribution, we have applied a homogenization process to the velocity potential acoustic wave equation. This allowed us to derive the actual laboratory realization of acoustic metamaterials exhibiting the effective properties prescribed by ATA. As an example, we designed a multilayer structure able to cloak the acoustic velocity potential. In addition, the analysis of several examples revealed an important difference in the way in which the results of STA and ATA should be implemented in a real device. In particular it appears clear that STA can be used only to design isobaric systems whereas ATA can be used only to design globally barotropic ones. Such conclusion depends critically on the assumptions on the thermodynamical properties of the fluid at the base of the derivation of the acoustic equations and it points towards new experimental approaches to the construction of acoustic metafluids.

Finally, it is worth mentioning that the work on ATA developed during this project has led to three research articles [20, 64, 65].

4.2 Analogue transformation elasticity

Let us summarize our findings about transformation elasticity:

- From our discussion it is immediate to realize that the scope of transformational elasticity is vastly broader than originally expected. One can implement different tensorial transformation laws, using three-dimensional (spatial) and four-dimensional (spacetime) tensors. One can even implement non-tensorial transformation laws completely or just for some particular sectors.

- On the one hand, we are still in need of a systematic treatment of the very essentials of the transformation techniques when applied to elasticity. It goes far beyond the form-invariant transformational properties of a physical system of equations as happens cleanly in optics, or as we have previously extended to the case of acoustics.

- On the other hand, it is essential to focus on specific examples of elastic systems and behaviours one would like to attain (a specific device), and then check which of the transformation laws gives the more practical or realistic implementation in the laboratory.

- The implementation of spacetime transformations would require the temporal control of some of the properties of the metamaterial system so as to lead to (apparent) flows in the metamaterial.

These are some of the topics we are analyzing at present and that eventually will be spin-offs of this exploratory work.

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