Triple Spacetime Metamaterials:

A generalization of transformation media.

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Maxwell's equations for linear media

Maxwell's equations in general coordinates

$$\nabla_i B^i = 0 \, , \qquad \nabla_0 B^i + \epsilon^{ijk}\partial_j E_k = 0 \, ,$$
$$\nabla_i \mathcal{D}^i = \rho \, , \qquad \epsilon^{ijk}\partial_j \mathcal{H}_k - \nabla_0 \mathcal{D}^i = j^i \, ,$$

are considered with a linear, nondispersive constitutive relation

$$\mathcal{D}^i = \epsilon^{ij} E_j + \kappa^{ij} \mathcal{H}_j \, ,$$
$$B^i = \mu^{ij} \mathcal{H}_j + \xi^{ij} E_j \, .$$

Symmetries and invariant transformations

Maxwell's theory is invariant under diffeomorphisms, locally coordinate transformations

$$x^\mu \Longrightarrow \bar{x}^\mu(x)$$

but

- two sets of equations

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \,, \qquad D_\nu H^{\mu\nu} = -J^\mu$$

  are invariant under separate diffeomorphisms;
- constitutive relation is not invariant under this manipulation.

Invariant transformation of the equations of motions, but no symmetry.

Generalization of transformation media

$\bar{x}^\mu = (\bar{t}, \vec{\bar{x}})$, $\bar{g}_{\mu\nu}$

$(\vec{\bar{\bar{E}}}, \vec{\bar{\bar{B}}})$

diffeomorphism I

re-interpretation

$x^\mu = (t, \vec{x})$

$g_{\mu\nu}$

$(\vec{E}, \vec{B})$

$(\vec{\mathcal{D}}, \vec{\mathcal{H}})$

$x^\mu = (t, \vec{x})$

$g_{\mu\nu}$

$(\vec{\tilde{E}}, \vec{\tilde{B}})$

$(\vec{\tilde{\mathcal{D}}}, \vec{\tilde{\mathcal{H}}})$

diffeomorphism II

re-interpretation

$\bar{\bar{x}}^\mu = (\bar{\bar{t}}, \vec{\bar{\bar{x}}})$, $\bar{\bar{g}}_{\mu\nu}$

$(\vec{\bar{\bar{\mathcal{D}}}}, \vec{\bar{\bar{\mathcal{H}}}})$

General constitutive relation

$$\tilde{\mathcal{D}}^i = -\bar{s}\frac{\sqrt{-\bar{\bar{g}}}}{\sqrt{\gamma}\, g_{\bar{0}\bar{\bar{0}}}} g^{\bar{\bar{i}}\bar{j}}\tilde{E}_j - \bar{s}\bar{\bar{s}}\frac{\sqrt{-\bar{g}}\sqrt{-\bar{\bar{g}}}}{\gamma\, g_{\bar{0}\bar{\bar{0}}}} g^{\bar{\bar{i}}\bar{k}} g^{\bar{\bar{0}}\bar{l}} \epsilon_{klm} g^{\bar{\bar{m}}\bar{j}} \tilde{\mathcal{H}}_j \ ,$$

$$\tilde{B}^i = -\bar{\bar{s}}\frac{\sqrt{-\bar{g}}}{\sqrt{\gamma}\, g_{\bar{0}\bar{\bar{0}}}} g^{\bar{i}\bar{\bar{j}}}\tilde{\mathcal{H}}_j + \bar{s}\bar{\bar{s}}\frac{\sqrt{-\bar{g}}\sqrt{-\bar{\bar{g}}}}{\gamma\, g_{\bar{0}\bar{\bar{0}}}} g^{\bar{i}\bar{\bar{k}}} \epsilon_{klm} g^{\bar{\bar{l}}\bar{0}} g^{\bar{\bar{m}}\bar{j}} \tilde{E}_j \ ,$$

with

$$g^{\bar{\bar{\mu}}\bar{\nu}} = \frac{\partial \bar{\bar{x}}^\mu}{\partial x^\rho}\frac{\partial \bar{x}^\nu}{\partial x^\sigma} g^{\rho\sigma} = \bar{\bar{g}}^{\mu\rho}\frac{\partial \bar{x}^\nu}{\partial \bar{\bar{x}}^\rho} = \frac{\partial \bar{\bar{x}}^\mu}{\partial \bar{x}^\rho}\bar{g}^{\rho\nu} \ ,$$

$\bar{s}$ change of orientation in $x^\mu \rightarrow \bar{x}^\mu$,

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Characteristics of generalized transformations

- $\bar{\bar{s}}\sqrt{-\bar{\bar{g}}}\mu^{ij} = \bar{s}\sqrt{-\bar{g}}\epsilon^{ji}$.
- ϵ^{ij} and μ^{ij} not symmetric if $\frac{\partial \bar{x}^{\mu}}{\partial \bar{\bar{x}}^{\nu}}$ not symmetric,
  $\Rightarrow$ non-reciprocal materials, e.g. $\bar{x} = x - z$, $\bar{\bar{x}} = x + z$.
- Signs of eigenvalues of ϵ^{ij} can be chosen freely,
  $\Rightarrow$ indefinite materials, e.g. $\bar{x} = -x$, $\bar{\bar{x}} = x$.
- Relative sign between eigenvalues of ϵ^{ij} and μ^{ij} not fixed.
  $\Rightarrow$ evanescent waves for $\bar{t} = -t$ but $\bar{\bar{t}} = t$.
- Bi-anisotropic materials if time maps non-trivially,

$$g^{\bar{\bar{0}}\bar{i}} \neq 0 \,, \qquad g^{\bar{0}\bar{\bar{i}}} \neq 0 \,.$$

κ^{ij} and ξ^{ij} need not be anti-symmetric.