Internal Mesh optimisation and Runge-Kutta collocation in a direct transcription method applied to interplanetary missions

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Abstract

Mesh refinment techniques are quite popular with collocation methods. These *a posteriori* adaptive methods, though, are unable to properly describe the optimal controls if a switching structure is present. An internal mesh optimisation strategy is here described that is able to solve this problem by modifing the collocation points positions and creating zero width intervals exactly in correspondance to the switching points. This leads to an algorithm that is able to properly describe the optimal control discontinuities by creating a mesh that has right nodes overlapped with left nodes in correspondance to the switching points.

Introduction

Direct transcription methods are used to perform optimisations in many different fields. Their application to aerospace problems has been quite popular in the past decades, and a number of dedicated professional softwares have been developed that are based upon this technique. Direct transcription methods [5] are usually based either on pseudospectral collocation methods (see Elnagar et al.[4] or Vasile [10]) or on simpler integration techniques (see Betts [2]). In both cases an infinite number of nodes is required to converge to the solution problems in which piecewise continuous control laws are optimal. This problem is connected with the inherent impossibility of properly describing discontinuities with these widely spread collocation methods. Placing one node on the switching point is not an answer even if it may increase the accuracy of the integration (see Paul [7]). In this paper a method is proposed that is able to move the collocation points creating zero width intervals around the switching points. The two overlapped nodes (called left node and right node) describe the discontinuity of the optimal solution with a high accuracy. The technique reveals to be able to locate the exact switching structure of the optimal control. This leads to a method that converges to the exact solution (in simpler problems) with a finite number of nodes, and that improves the prediction capabilities of existing techniques being able to better represent the discontinuities of the optimal control law.

Setting some benchmarks

In this paragraph we face the synthesis of three different optimal time controls by applying the methodology based on Pontryagin Maximum Principle and described in [9]. We chose the three problems in order to obtain feedbacks in the class of bang-bang controls (linear systems always return time optimal solutions of this type) and of bang-off-bang controls (non linearities are needed), and in order to represent problems of interest to the aerospace community. The main goal of solving these three problems is that of setting a benchmark for the numerical algorithms that are aimed at revealing the switching structure of an optimisation problem.

Problem 1 (classical):

$$\begin{cases} \dot{x} = v \\ \dot{v} = u \\ x(0) = x_0, v(0) = v_0 \\ x(t_f) = 0, v(t_f) = 0 \\ \min\limits_{\|u\| \leq 1} t_f \end{cases} \tag{1}$$

where the admissible controls live in the functional space of piecewise continuous functions with $|u| \leq 1$. We build the Hamiltonian function $\mathcal{H}$ introducing the co-states ψ:

$$\mathcal{H} = \psi_1 v + \psi_2 u$$

The co-states are defined by the equations:

$$\dot{\psi}_1 = 0$$
$$\dot{\psi}_2 = \psi_1$$

that may be solved straight-forward (as it is always the case for time optimal control of linear systems) returning:

$$\psi_1 = c_1$$
$$\psi_2 = c_1 t + c_2$$

Applying Pontryagin maximum principle we get that:

$$u = \text{sgn}(c_1 t + c_2)$$

as a consequence the optimal trajectory may switch one time only and the optimal feedback and state trajectories (see Figure 1) are obtained by studying the sliding surface (in this case coincides with the switching surface). The time optimal control is therefore bang-bang as expected. The sliding curve has equation:

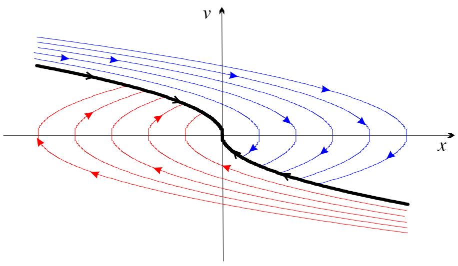

Figure 1: Feedback synthesis for the classical problem 1.

$$\tilde{v}(x) = \begin{cases} -\sqrt{2x} & \text{if} x > 0 \\ \sqrt{-2x} & \text{if} x < 0 \end{cases}$$

whereas the switch time t^* has expression:

$$t^* = \begin{cases} \sqrt{\frac{1}{2}v_0^2 - x_0} - v_0 & \text{if} v_0 < \tilde{v}(x_0) \\ \sqrt{\frac{1}{2}v_0^2 + x_0} + v_0 & \text{if} v_0 > \tilde{v}(x_0) \end{cases} \quad (2)$$

Finally one is even able to obtain an analytical expression for the optimal time t_f as a function of the initial conditions:

$$t_f = \begin{cases} 2\sqrt{\frac{1}{2}v_0^2 - x_0} - v_0 & \text{if} v_0 < \tilde{v}(x_0), (+1, -1) \\ 2\sqrt{\frac{1}{2}v_0^2 + x_0} + v_0 & \text{if} v_0 > \tilde{v}(x_0), (-1, +1) \end{cases} \quad (3)$$

Problem 2 (Nutation control):

$$\begin{cases} \dot{p} = u_p + \tilde{\Omega} q \\ \dot{q} = u_q - \tilde{\Omega} p \\ p(0) = p_0, q(0) = q_0 \\ p(t_f) = 0, q(t_f) = 0 \\ \min\limits_{\|u_p\| \leq U_p, \|u_q\| \leq U_q} t_f \end{cases} \quad (4)$$

This problem arises in connection with the time optimal control of the nutation angle of a spinning satellite, being a linear problem the optimal feedback may be written in a closed form and will be bang-bang. We start building the Hamiltonian function $\mathcal{H}$ introducing the co-states ψ:

$$\mathcal{H} = (u_p + \tilde{\Omega} q)\psi_1 + (u_q - \tilde{\Omega} p)\psi_2$$

The co-states are defined by the equations:

$$\dot{\psi}_1 = \tilde{\Omega}\psi_2$$
$$\dot{\psi}_2 = -\tilde{\Omega}\psi_1$$

that may be solved returning:

$$\psi_1 = A \sin(\tilde{\Omega} t + \alpha)$$
$$\psi_2 = A \cos(\tilde{\Omega} t + \alpha)$$

The optimal feedback would therefore take the form:

$$u_p = \text{sgn}\left[A \sin(\tilde{\Omega} t + \alpha)\right]$$
$$u_q = \text{sgn}\left[A cos(\tilde{\Omega} t + \alpha)\right]$$

This last equation tells us that the time optimal nutation control of a spinning gyrostat is a bang-bang control with switch frequency equal to $\frac{\tilde{\Omega}}{2\pi}$. To actually get the complete feedback we need to study the sliding and the switching surfaces and the optimal state trajectories (see figure 2) which are easily seen

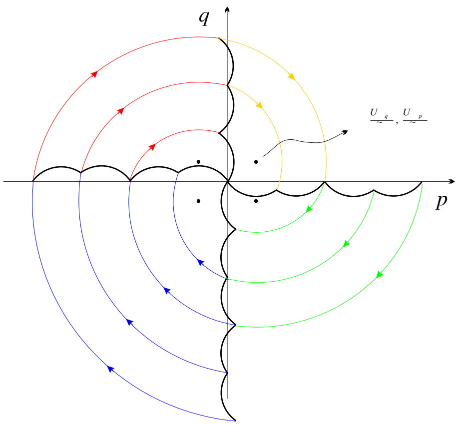

Figure 2: Time optimal state trajectories for the nutation control problem.

to be circles centered in $(\pm\frac{Up}{\Omega}, \pm\frac{Uq}{\Omega})$ Writing down the expressions for the sliding and the switching surface is now a matter of algebraic exercise, the same being valid for the optimal time expression.

Problem 3 (Dubins' car):

$$\begin{cases} \dot{x} = \cos\theta \\ \dot{y} = \sin\theta \\ \dot{\theta} = u \\ x(t_f) = 0, y(t_f) = 0, \theta(t_f) = \begin{cases} \text{any} \\ \tilde{\theta} \end{cases} \\ \min\limits_{\|u\|\leq 1} t_f \end{cases} \tag{5}$$

This problem has been studied in connection with air traffic management, differential games, hybrid control systems. It has the advantage of being rather simple and non linear. It is interesting to see how the application of the maximum principle leads us to the synthesis of an optimal feedback through a series of wit reasonings. The Hamiltonian has, in this case, the form:

$$\mathcal{H} = \cos\theta\psi_1 + \sin\theta\psi_2 + u\psi_3$$

where the co-states ψ are defined by the differential system:

$$\begin{cases} \dot{\psi}_1 = 0 \\ \dot{\psi}_2 = 0 \\ \dot{\psi}_3 = \psi_2\cos\theta - \psi_1\sin\theta \end{cases}$$

that has the solution:

$$\begin{cases} \psi_1 = c_1 \\ \psi_2 = c_2 \\ \psi_3 = c_3 + \int_{t_0}^{t}(c_2\cos\theta - c_1\sin\theta)dt \end{cases}$$

The Maximum Principle tells us that the optimal feedback is ± 1 whenever ψ_3 is not zero over a finite time period. In this non linear problem this is in-fact possible with $u = 0$, in which case $\theta = \text{cost.} = \bar{\theta}$ and $c_3 = 0$, $\frac{c_2}{c_1} = \frac{\sin\bar{\theta}}{\cos\theta}$. On the other hand a switch is possible whenever $\psi_3 = 0$, that is during an off phase, or in t_{sw} being

$$c_3 + \int_{t_0}^{t_{sw}}(c_2\cos\theta^* - c_1\sin\theta^*)dt = 0$$

where θ^* is the optimal trajectory. This last equation is an algebraic relation between the three constants c_1, c_2 and c_3. In the case in which we consider the problem with $\theta(t_f) = \text{any}$ we must also take into account the transversality condition that is in this case written as $\psi_3(t_f) = 0$ (an algebraic relation between c_1, c_2, c_3). We may therefore conclude (by counting the relations between the constants) that the time optimal control for the Dubins' car problem with free final θ admits one only switch and may be bang-bang, off-bang or bang-off depending on the initial conditions. The other case (the fixed final θ) admits two switches and may be bang-off-bang, off-bang-bang etc., etc. Any trajectory that satisfies these conditions, and the boundary conditions, is allowed by Pontryagin Maximum Principle. In Figure 3 some allowed optima are shown. Write some feedback by using the sole Maximum Principle is therefore not possible, some kind of other information is necessary. In this simple case we simply observe that the bang-bang solution is never optimal and that we have to go either for the bang-off solution (turning in the direction that allows for a smaller part of the circle to be drawn), or for the off-bang solution, when we are inside that zone of the state space which would

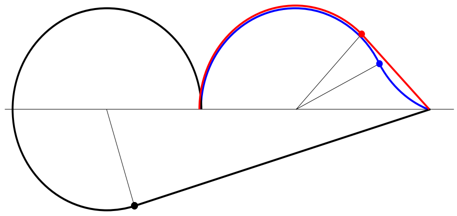

Figure 3: Possible optima allowed by the Maximum Principle. The global optimal strategy is, of course, the red one.

not allow to reach the origin with a bang-off strategy (half of this zone is shown in figure 4, the other half is obtained by rotation by π) The state space is there-

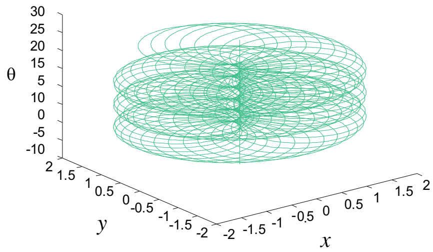

Figure 4: Zone III in the Dubin's car optimal feedback (free final angle): off-bang solution.

fore divided into three zones in which the control is respectively $u = 1$, $u = -1$ (circular x, y trajectory) and $u = 0$ (rectilinear x, y trajectory). Between zone I and zone II a switching surface exists where $u = 0$ (see Figure 5). To be able to see this three dimensional time optimal synthesis an intersection is shown between the state feedback and the plane $\theta = \frac{\pi}{4}$ in Fiugre 6. In the case of fixed final θ the feedback synthesis is even more complicated from a geometrical point of view, the reader may though realize that far from the origin the bang-off-bang strategy is the time optimal one.

From the three problems solved above one immediately realizes how the time-optimal control has difficult geometrical properties even in the simplest cases described by linear systems. The introduction of nonlinearities make the use of the Maximum Principle

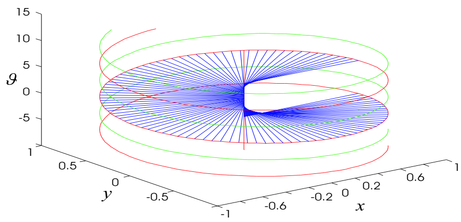

Figure 5: Zone I-II border in the Dubin's car optimal feedback (free final angle).

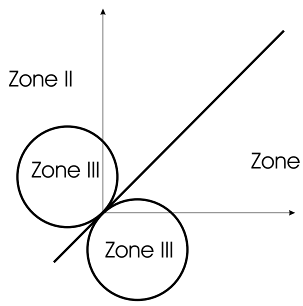

Figure 6: State space zones in the Dubins' car problem with free final angle and for $\theta = \frac{\pi}{4}$

quite difficult and introduces the need of some kind of other observations in order to locate the global optima. Several local optima are created by the introduction of the simplest non-linearities, the optimal strategy gets quite complex and non continuously dependent on the state. For optimal problems in which the objective function is not the time the situation gets even more complicated, with infinite switch times being possible even in the simple classical problem 1 (Fuller effect see [8])

Mesh Optimisation

In the simple cases shown in the previous paragraph, the optimal trajectories are found by means of a feedback synthesis. It has been underlined how Pontryagin Maximum Principle is satisfied by also non optimal trajectories that constitute local minima and that it is therefore necessary, in the feedback synthesis, to argue somehow what switching curve has to be chosen. An analogous role is played by the initial guess selection in the numerical algorithms aimed at solving an optimization problem. ESA Advanced Concepts Team, under the Ariadna scheme, is coordinating and funding some studies on the global search of feasible near optimal solutions aiming at classifying the complexity of the optimization problems related to mission analysis issues and at locating the most suitable global approach for each of them. A particular attention is given to multiple gravity assists trajectories and to low thrust propulsion arcs (solar sails, solar electric propulsion, nuclear electric propulsion) and different global techniques are assessed. The solution found by a space search is then usually passed to a local optimiser that refines it to the nearest real minima. Within the context of interplanetary trajectories a commonly used approach to carry out the local optimisation is the direct transcription method. This consists in substituting the optimal control problem with an NLP problem and in solving this instead. This "substitution" has to be done by taking care that the discretized optimal control problem converges to the solution of its infinite dimensional relative. There are little or no results in literature regarding proofs of the convergence of even the most popular algorithms. As it has been pointed out in [6] some simple examples may easily be presented that shows how even methods that usually converge in a well defined way may show strange behaviors or may not converge at all. The opposite phenomenon (convergence of nonconvergent algorithms) has also been observed and an explanation appeared in a recent paper by Betts et al. [3]. The convergence of any direct transcription algorithm may therefore not be inferred from standard analysis of the underlying integration scheme used, but only from an *ad hoc* analysis of the particular transcription method. All these comments are applicable when the optimal control is a continuous function or when the switching structure of the optimal strategy is known *a priori*. If the transcription method has also to locate the switching structure of the optimal strategy, things get even more complicated and statements on the convergence are almost impossible to introduce.

An important issue in all direct transcription algorithm is where to collocate the grid points, that is the time mesh design. This is usually done by starting with an equally spaced mesh (or some other initial mesh decided by the user on the basis of his knowledge of the particular problem treated) taking a look at the solution and applying some mesh refinement strategy (automated or user based). The optimization is then run again on the new mesh. The idea on which this work is based is that the mesh refinement might be done during the main optimization by introducing a multi-objective problem in which the main goal is pursued together with some other objective that, in practical terms, optimize the mesh as well. This does not include, of course, the possibility of augmenting the number of nodes in the mesh, but it leads to an increased accuracy in the prediction of the states and of the controls, or in the individuation of the eventual switching structure, so that the same accuracy might be reached by a smaller dimension problem.

Let us first describe the numerical transcription method we will use throughout the paper. The method is quite standard, see for example [1], excepts in the defects definition. Let the time scale be divided into N points t_k, $k = 1..N$. The variables considered in the Non Linear Programming (NLP) problem are:

$$\mathbf{z} = [\mathbf{x}_k, \mathbf{u}_k, \mathbf{u}_{mk}, t_k], \quad k = 1..N \tag{6}$$

where $\mathbf{x}_k$ is the state at time t_k, $\mathbf{u}_k$ is the control at time t_k and u_{mk} is the control at time $\frac{t_k+t_{k+1}}{2}$. The continuous constrains of the Optimal Control Problem (OCP) have now to be transcribed into some algebraic constrains. This is done by using a fourth order Runge Kutta formula exploiting the states at the points t_k and the controls at the point t_k and $\frac{t_k+t_{k+1}}{2}$. The defects have been written as:

$$\zeta_k = \mathbf{x}_{k+1} - \mathbf{x}_k - \frac{h_k}{6}[\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4]$$

where:

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) \\
\mathbf{k}_2 &= \mathbf{f}(\mathbf{x}_k + \tfrac{\mathbf{k}_1}{2}, \mathbf{u}_{mk}) \\
\mathbf{k}_3 &= \mathbf{f}(\mathbf{x}_k + \tfrac{\mathbf{k}_2}{2}, \mathbf{u}_{mk}) \\
\mathbf{k}_4 &= \mathbf{f}(\mathbf{x}_k + \mathbf{k}_3, \mathbf{u}_{k+1})
\end{aligned}$$

Note that with this method the control mesh is finer than the state mesh and that the method is similar to the one preferred by Betts [3] and based on Trapezoidal or Hermite-Simpson collocations. Once the problem has been transcribed, the NLP solver is free to choose the mesh grid. In order to drive its choice towards meshes that improve the quality of the prediction, the objective function cannot be simply in the Bolza form, otherwise a mesh would be found that numerically underestimates as much as possible the solution with respect to the value we want to minimize. A good strategy could be that of trying to minimize, together with the objective function, the error introduced by the numerical integration scheme. This would have, as an effect, to concentrate the nodes where high gradients of the dynamic exist (due to natural or forced terms). In some control problems this might be a viable strategy and is therefore here briefly discussed. An estimate of the

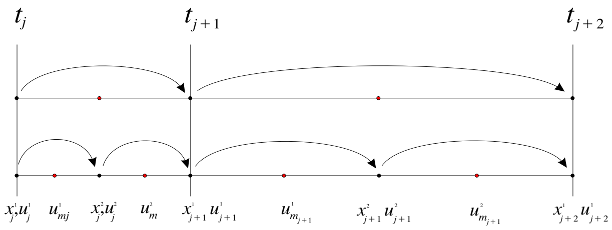

Figure 7: Mesh shape in a numerical error control strategy

numerical error introduced by the discretization may be done by using the techniques derived from classical adaptive stepsize methods. With this respect we may choose between step-doubling techniques and embedded Runge-Kutta techniques (originally developed by Fehlberg). These last techniques require less functions evaluations, but are not suitable to our purposes as our dynamic is not known over the entire interval, but only in some discrete points (the control is in-fact an unknown). We remain with the standard step-doubling techniques. The implementation of these technique requires a modification of the definition of our unknowns, should we use the variables showed in 6 we would not be able to evaluate the Runge-Kutta formula without having to estimate the state or the control in some point not belonging to the grid. In order to avoid this problem the grid has to be designed as shown in figure 7. Between two free time nodes there are three equally spaced nodes in which the the controls are considered as unknowns. Moreover, in order to apply the Runge-Kutta algorithm, the state has to be considered as unknown in the central auxiliary node. The variables considered in the resultant Non Linear Programming (NLP) problem are:

$$\mathbf{z} = [t_k, \mathbf{x}^1_k, \mathbf{u}^1_k, \mathbf{u}^1_{mk}, \mathbf{x}^2_k, \mathbf{u}^2_k, \mathbf{u}^2_{mk}], \quad k = 1..N$$

This approach allows for some control on the mesh at the cost of increasing the problem dimension, moreover, when discontinuities on the control are present, (for example in minimum mass or minimum time problems) it fails (as any other algorithm) to locate the exact switching point of the control. This last issue is fundamental and it is maybe the most important think that might be asked to an internal mesh optimisation. As a switching point is a point in which the control is discontinuous we may require the optimizer to try to minimize, in addition to our main goal, the sum of all the time intervals weighted with the control discontinuity. This should lead the optimizer to overlap two mesh main points in the discontinuity point creating a left and right part of the solution. By assuming as variables the ones appearing in Eq.(6) we write the objective function in the form:

$$J = J + c \sum_{i=1}^{N-1} \Delta u_i \Delta t_i \tag{7}$$

where $\Delta u_i = u_{i+1} - u_i$ and $\Delta t_i = t_{i+1} - t_i$. The concern is that now the optimizer will try to pursue two different goals and a careful choice of the constant c has to be made. Moreover the basin of attraction of local minima might be modified so that convergence from a given initial solution may not happen any more. These issues have to be discussed on the basis of the particular problem treated.

We apply the technique described above to the classic *problem 1* in order to show its potentials. First we solve the problem by using a standard approach. We here used the Runge-Kutta collocation with an equally-spaced grid, but the conclusions we will draw apply to a generic algorithm. Without loss of generality we show here the case of initial conditions $x = 3$, $v = 2$. From Eq.(2, 3) we may evaluate, in this case, a switch time $t^* = 2 + \sqrt{5}$ and an optimum time of $t_f = 2(1 + \sqrt{5})$. The standard approach returns, with $N = 4$ nodes, the results visualized in figure 8. The main nodes are shown together with the auxiliary points where only the control is considered as unknown. The final time returned is a good approximation of the exact value, whereas the switch time is not returned by the numerical solution and may only be estimated *a posteriori* introducing some assumption or numerical estimator. The problem is encountered also with Pseudospectral collocation methods as the points are there collocated in such a way that nothing prevents the control discontinuity to happen in the middle of a large interval. Let us now take a look at the results returned by the use of the objective function defined by Eq.(7). These are visualized in figure 9. The constant c has been set to be $c = 1.1$. The number of nodes is still $N = 4$, but this time the NLP solver (based on SQP) is able to move the mesh points so that it manages to make the switch happen in an interval of zero width. As a consequence two of the four main nodes are overlapped in the time grid. This allows to determine the exact location of the switch point and allows the underlying integration scheme to increase its accuracy. In this simple case (linear dynamic) the values returned by using the mesh optimisation are exact! This result could not be achieved by any other direct collocation technique that the authors are aware of. Numerical simulations on the nutation control problem and on the Dubin's car problem showed that the technique is applicable to other cases with encouraging results and that it is therefore worth trying with the complex interplanetary transfer problem. In this last case the internal mesh optimisation technique described should be able to determine the exact location of the ballistic and of the propelled trajectory arcs typical in minimum mass problems.

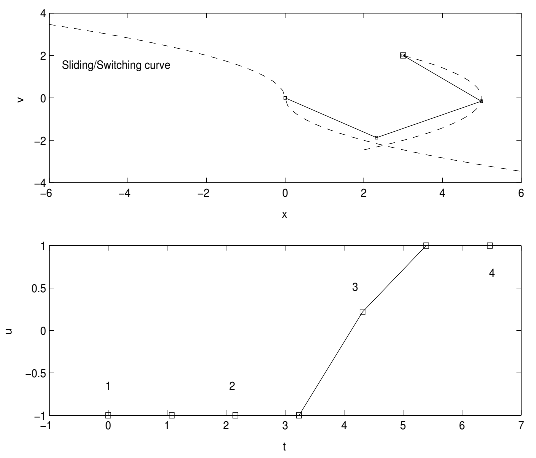

Figure 8: Sample of the results returned by using a classical approach (4 Nodes)

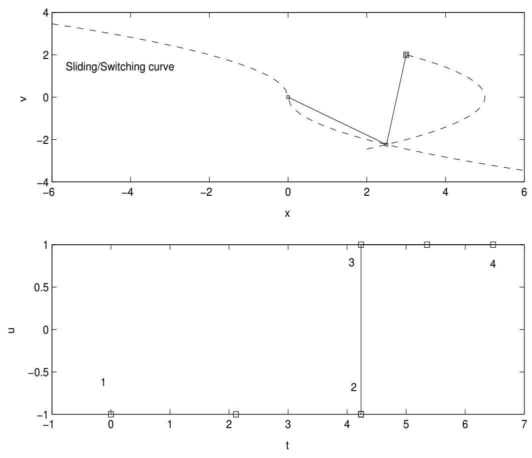

Figure 9: Sample of the results returned by optimizing the mesh (4 Nodes)

The application to interplanetary trajectories

In this section we will apply the proposed approach to a simple case of interplanetary trajectory optimisation. We consider a case in which only one dynamical phase is present, a simple low-thrust transfer from the Earth to Mars of a massive spacecraft (10000 kg) that has to be captured by the arrival planet (arrival C3 set to be zero, departure C3 set to be less than 10 km^2/sec^2). The arrival mass is maximized by assuming a maximum thrusting capability of roughly $1N$ with a specific impulse equal to $4000sec.$. The values of this optimization have been motivated by recent ACT researches (see [11]). Cartesian coordinates are chosen to model the dynamic so that the following system of equations is used:

$$\begin{cases} \dot{\vec{\mathbf{v}}} + \mu \frac{\vec{\mathbf{r}}}{r^3} = \frac{\vec{\mathbf{u}}}{M} \\ \dot{\vec{\mathbf{r}}} = \vec{\mathbf{v}} \\ \dot{m} = -\frac{u}{I_{sp} g_0} \end{cases}$$

The maximum thrust dependency upon solar power available has been not taken into account and departure and arrival dates have been optimized too. For this particular problem a standard direct transcription approach returns the results shown in figure 10. The in-plane component of the thrust are plotted together with the total thrust. The control has a bang-off structure that is captured by the transcription mehod, even though the switching structure cannot be revealed perfectly as the nodes are not free to move. If we now allow the nodes to move and

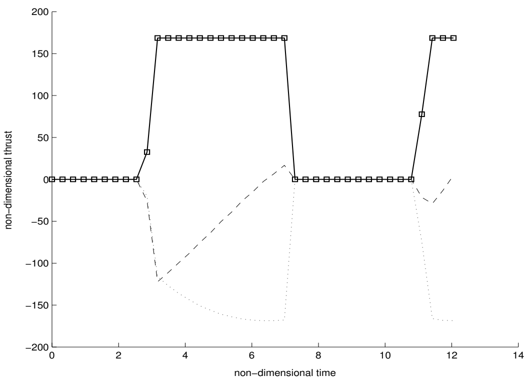

Figure 10: Optimal control law for an Earth-Mars transfer (no mesh optimisation). The dashed and dotted curves represent the in-plane components of the optimal thrust. The control mesh is also shown.

modify the objective function according to eq.(7) we get the new solution shown in figure 11. The resulting trajectory is shown in figure 12. As it happened with the simple dynamics that have been proposed at the beginning of the work, also in this more complex

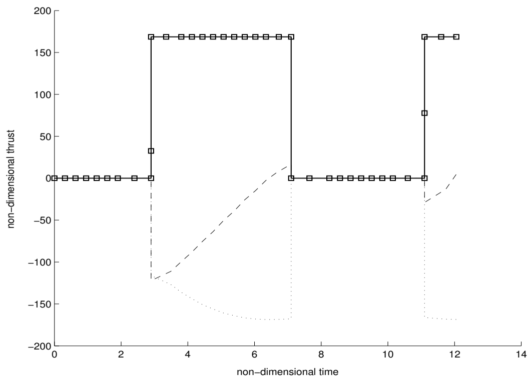

Figure 11: Optimal control law for an Earth-Mars transfer (with mesh optimisation). The dashed and dotted curves represent the in-plane components of the optimal thrust. The control mesh is also shown.

case, the internal mesh optimization allows for a better solution in terms of switching structure location and resulting solution accuracy. The price to pay is of course a decreased convergence speed of the NLP solver. When all the nodes are left free to vary the solver sometimes moves them far away from the original location (in the off phase Δu is costantly zero). A solution is to leave free only the nodes that are immediately before and after a located discontinuity, but this requires a preliminary run of a fixed mesh optimisation.

Conclusions

A novel technique to optimise the mesh in a direct transcription method is proposed and assessed. The mesh is optimised in order to improve the accuracy in the representation of a piecewise discontinuous function. This is obtained by creating a fictitious minimum in correspondance of a mesh that has two consecutive nodes overlapped over the control switching point (unkown *a priori*). This creates a zero width interval that has a right and a left node improving the representation of the optimal discontiuous controls. The technique, applied first to problems with a simple dynamic and than to a more complex Earth-

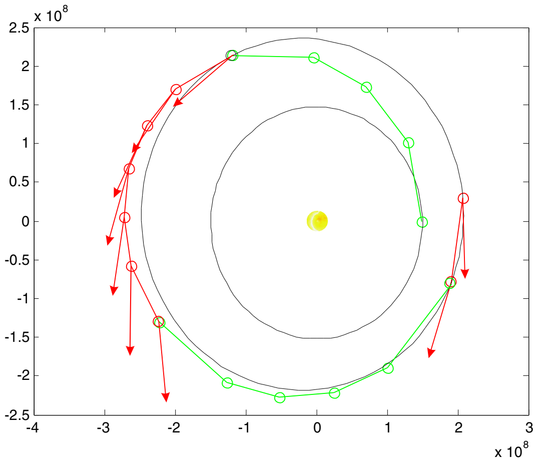

Figure 12: Optimal mass trajectory for an Earth-Mars transfer (note the different length of the integration steps around the swithing points).

Mars transfer is shown to be able to adapt the mesh grid and to require less collocation nodes to solve the problem with the same accuracy level.

Acknowledgments

The original research on this topic started in early February 2004 under the enlighting guide of Prof. Chiara Valente. She died some weeks after the abstract was submitted without having the chance to direct me also in this further research.

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