Command shaping for a flexible satellite platform controlled by advanced fly-wheels systems

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Abstract

Coping with flexibility is an important issue in space applications as it allows to design lighter structures and therefore to save considerable amounts of mass. A number of works have been published in the past showing how a careful design of the attitude control system may drastically improve the system response reducing the residual vibrations. In this paper we consider the coupling between the flexibility of a spacecraft equipped with Variable Speed Control Moment Gyros and its attitude control system, in the framework of command shaping techniques. The analysis is performed on the complete set of equation of motion written in an explicit form (and with a block diagonal mass matrix) describing a flexible platform and N wheels gimballed to it. The equations are written by standard multibody techniques. The wheels are Variable Speed Control Moment Gyroscopes (two degrees of freedom) and their control system is designed upon a Lyapunov based feedback relying upon a rigid body model. Whenever the structure is not stiff enough, flexibility degrades the performances of the controller that fails to track the desired history due to the induced vibrations. We therfore study the possibility of altering the tracking signal fed to the controller trying to get rid of the relevant frequencies notch filtering them in order to reduce vibrations. The results obtained for a test case attitude acquisition maneuver are presented and the benefit deriving from the command shaping technique evaluated and discussed.

1 Introduction

In recent years the advances achieved in spacecraft construction techniques have brought many researchers to investigate the dynamics of large and lightweight structures in space. Large solar arrays structures are being considered for a number of recent mission concepts. Flexible booms are already used in interplanetary missions (having RTGs, magnetometers or other payloads on their tip) and for stabilization purposes. Large space structures are continuously proposed in advanced concepts regarding the exploration strategy of our solar system (interplanetary gateways concepts). Therefore, in the future of space missions, the flexibility of structural elements will certainly become more and more important. On the other hand high agility demands impose on spacecraft designers the use of actuators that are capable to erogate a large amount of torque. Among the control devices able to generate torques the most commonly used today are flywheels. These wheels store angular momentum and exchange it with satellite platform main body causing spacecraft reorientation. Control Moment Gyros (CMG) and Variable Speed Control Moment Gyros (VSCMG) represent a particular group of flywheels able to exploit the amplification torque effect to exchange angular momentum with the platform in a more efficent way. The influence of structural vibrations on the attitude dynamic will be more and more meaningful as these new control devices will appear on board the modern satellites. In fact, since these new actuators are able to produce greater torques, they increase the effects due to the nonlinear coupling between the flexible and the attitude dynamics. A set of equations able to describe the dynamics of a flexible satellite equipped with advanced flywheels actuators could result in a great advantage in the design of control laws for such a system, especially when the requirements are high maneuvrability of the spacecraft and high precision of the final pointing. The equations of motion for this complex satellite can be derived according to multi-body analysis techniques [5, 9, 6]. Multibody dynamic has attracted the efforts of many researchers in the past years. Methods to obtain the set of decoupled non linear equations governing a generic holonomic system made of rigid and deformable bodies have been researched. Both assumed modes methods and finite element methods have been used to model the flexible degrees of freedom of the system. Newton-Euler approach, Lagrange approach and the more recent Kane [3] approach have been used to write the final equations governing the dynamic of the holonomic system.

Many works have also been written on the control of rigid spacecrafts equipped with any system of flywheels. Some definitive results have been achieved in this field. The work of Oh and Vadali [7] contains the full derivation of the equations for a Control Moment Gyros system and the development of two different Lyapunov based control laws based on gimbal rates and gimbal accelerations. In the work by Shaub et al. [8] Variable Speed Control Moment Gyros are considered and Tsiotras et al. [10] develop some energy tracking system for these devices.

In this paper both the results obtained in literature on the dynamics and on the control of such complex systems will be exploited in order to evaluate the performance of a flexible satellite that has to perform a rest-to rest angular maneuver. This paper augments and completes a previous work by the authors [1] in which the influence of structural vibrations upon control law's performances has been verified for VSCMG systems varying the input guidance shapes. In particular the system performance has been tested for different angular acquisition maneuvers in that paper. A numerical simulation campaign shows how, varying the total acquisition time of the tracking signal, it is possible to find the fastest real acquisition that the flexible satellite can accomplish. In the present work a command shaping technique is proposed that intends to eliminate the natural frequency of the system from the tracking signal exploiting bandstop filters. The results obtained are shown and the advances triggered using this approach are assessed and discussed.

2 Flexible spacecraft dynamical model

In order to derive the equations of motion for a flexible satellite equipped with a cluster of VSCMG a mixture of Newton-Euler approach and of Lagrange approach has been followed. The balance of momentum and the balance of the absolute angular momentum has been written using a standard Eulerian approach, whereas the remaining equations, those determining the dynamic of the flexible variables ε has been determined with the aid of a Lagrangian approach. The resulting set of equation of motion is given by the momentum balance equation

$$\begin{aligned}&\mathbf{f} = \\ &= M\mathbf{a} + \dot{\boldsymbol{\omega}} \times \sum \mathbf{L}_i\varepsilon_i + \sum_i \mathbf{L}_i\ddot{\varepsilon}_i + \\ &+\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \sum \mathbf{L}_i\varepsilon_i) + 2\boldsymbol{\omega} \times \sum \mathbf{L}_i\dot{\varepsilon}_i\end{aligned} \tag{1}$$

the angular momentum balance equation

$$\begin{aligned}&\mathbf{J}_T\dot{\boldsymbol{\omega}} + \textstyle\sum_j Y_{g_j}\ddot{\gamma}_j\mathbf{g}_j + \sum_j I^w_{s_j}\dot{\Omega}_j\mathbf{s}_j + \sum I^w_{s_j}\Omega_j\dot{\gamma}_j\mathbf{t}_j \\ &+ \textstyle\sum_j (Y_{s_j} - Y_{t_j})\dot{\gamma}_j(\mathbf{t}_j\mathbf{s}_j^T + \mathbf{s}_j\mathbf{t}_j^T)\boldsymbol{\omega} + \boldsymbol{\omega}^x\mathbf{J}_T\boldsymbol{\omega} + \\ &+ \textstyle\sum_j Y_{g_j}\dot{\gamma}_j\boldsymbol{\omega}^x\mathbf{g}_j + \sum_j I^w_{s_j}\Omega_j\boldsymbol{\omega}^x\mathbf{s}_j + \\ &+ \textstyle\sum_i \mathbf{L}_i\varepsilon_i \times \mathbf{a} + \sum_i \left(\mathbf{G}_i + \sum_k (\mathbf{H}_{ik} + \mathbf{H}_{ki})\varepsilon_k\right)\dot{\varepsilon}_i\boldsymbol{\omega} + \\ &+ \textstyle\sum_i \left(\mathbf{S}_i - \sum_k \mathbf{P}_{ik}\varepsilon_k\right)\ddot{\varepsilon}_i + \\ &+ \boldsymbol{\omega}^x \textstyle\sum_i \left(\mathbf{S}_i - \sum_k \mathbf{P}_{ik}\varepsilon_k\right)\dot{\varepsilon}_i = \mathbf{g}\end{aligned} \tag{2}$$

and the the flexible coordinates equations

$$\begin{aligned}&\rho\ddot{\varepsilon}_k + c_k\dot{\varepsilon}_k + k_k\varepsilon_k = \\ &= -\mathbf{L}_k \cdot \mathbf{a} - (\mathbf{S}_k - \textstyle\sum \mathbf{P}_{kj}\varepsilon_j) \cdot \dot{\boldsymbol{\omega}} + 2\boldsymbol{\omega} \cdot \sum \mathbf{P}_{kj}\dot{\varepsilon}_j \\ &+ \tfrac{1}{2}\boldsymbol{\omega} \cdot \left(\mathbf{G}_k + \textstyle\sum (\mathbf{H}_{kj} + \mathbf{H}_{jk})\varepsilon_j\right)\boldsymbol{\omega} + Q_k \\ &k = 1, .., N\end{aligned} \tag{3}$$

An explication of all the symbols appearing in this equations is given in the appendix. The detailed derivation of such a set of equation can be found in [2] together with an implementation scheme that brings the set of coupled equations in a explicit form with block-diagonal mass matrix.

3 Control law definition

In this section the derivation of the control feedback considered in this work will be presented. Such a control steering law is designed for a rigid spacecraft equipped with a cluster of VSCMG and it will be applied to a model developed for a flexible spacecraft. The equation of motion upon which the control law is designed is

$$\begin{aligned}&\mathbf{J}_T\dot{\boldsymbol{\omega}} + \textstyle\sum_k Y_{g_k}\ddot{\gamma}_k\mathbf{g}_k + \sum_k I^w_{s_k}\dot{\Omega}_k\mathbf{s}_k + \sum_k I^w_{s_k}\Omega_k\dot{\gamma}_k\mathbf{t}_k \\ &+ \textstyle\sum_k (Y_{s_k} - Y_{t_k})\dot{\gamma}_k(\mathbf{t}_k\mathbf{s}_k^T + \mathbf{s}_k\mathbf{t}_k^T)\boldsymbol{\omega} + \boldsymbol{\omega}^x\mathbf{J}_T\boldsymbol{\omega} + \\ &+ \textstyle\sum_k Y_{g_k}\dot{\gamma}_k\boldsymbol{\omega}^x\mathbf{g}_k + \sum_k I^w_{s_k}\Omega_k\boldsymbol{\omega}^x\mathbf{s}_k = \mathbf{g}\end{aligned} \tag{4}$$

It is easy to see that Eq.4 is a particular case of Eq.2 and it can be obtained from that dropping the terms in which flexibility appears. Kinematics relations given in terms of Modified Rodriguez Parameters (MRP) have to be added to Eq.(4) in order to complete the set of motion equations for the multi-body satellite

$$\dot{\mathbf{q}} = \mathbf{G}(\mathbf{q})\mathbf{q}. \tag{5}$$

The whole set of equation can be written in the compact form

$$\mathbf{f}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$$

where

$$\begin{aligned} \mathbf{x} &= [\boldsymbol{\omega}_e, \mathbf{q}_e] \\ \mathbf{u} &= [\dot{\gamma}_j, \dot{\Omega}_j] \end{aligned} \tag{6}$$

are the configuration and control vectors respectively. Moreover, in Eq.(6), $\mathbf{q}_e$ and $\boldsymbol{\omega}_e$ represent respectively the error of the spacecraft attitude and angular velocity with respect to the tracking signal whereas $\dot{\gamma}_j, \dot{\Omega}_j$ are the $j-$th wheel angular rates with respect to the gimbal and spin axis, used as control variables.

Following the Lyapunov approach proposed in [7], the following relation between control and configuration parameters can be obtained

$$\begin{aligned} &\mathbf{B}\ddot{\gamma} + \mathbf{C}\dot{\gamma} + \mathbf{D}\dot{\Omega} = \\ &= \mathbf{K}\boldsymbol{\omega}_e + k_0\boldsymbol{\sigma}_e - \mathbf{J}_T\boldsymbol{\omega}_d - \boldsymbol{\omega}^x(\mathbf{J}_T\boldsymbol{\omega} + \\ &+ \sum I^w_{s_k}\mathbf{s}_k\boldsymbol{\Omega}_k) + \mathbf{g}_b = \mathbf{L}_{rm} \end{aligned} \tag{7}$$

where $\mathbf{L}_{rm}$ is the required torque and the complete expressions for $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ can be found in [8]. Eq. (7) is a set of 3 relations between $2n$ unknown quantities. A maximum constrained problem provides other equations and allows to obtain the final relation between control parameters $\mathbf{u}$ and system's coordinates $\mathbf{x}$ as follows

$$\mathbf{Q}\mathbf{u} = \mathbf{L}_{rm} \tag{8}$$

4 Spacecraft's test configuration

In the following the simple geometrical configuration considered as a test case is presented. This configuration has been chosen in order to obtain from an analytic approach the shape functions of the sole flexible appendage. Whenever a more complex case is under consideration only the integral quantities describing the deformation of the appendages have to be loaded in the model. These quantities can be obtained using FEM techniques and have to be fed to the decoupled equations for the various $\mathbf{J}$, $\mathbf{G}$, $\mathbf{H}_{ij}$ parameters. The satellite considered in this work as a testcase is shown in figure 1. It is made up of a central rigid hub in which a cluster of VSCMG in a piramidal configuration is embedded. Attached to the main structure a $6m$ long flexible boom is considered. The body frame $\mathcal{F}_b$ is centered in the center of mass G of the undeformed satellite and has the x axis aligned with the undeformed boom. The boom root has a distance of .6 meters from the point G. For a selected ratio of the tip

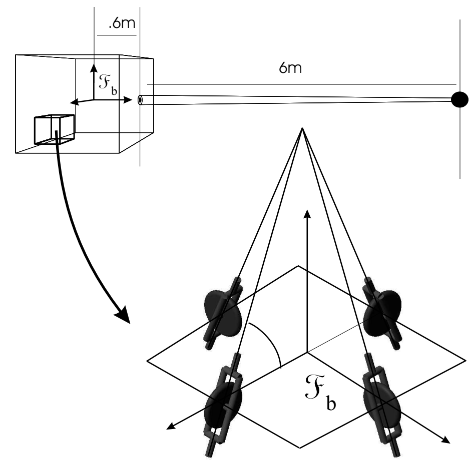

Figure 1: Geometrical configuration considered

mass to the beam mass of .75 and by assuming as flexible stiffness $EI = 12Nm^2$, the translational and rotational participation factors may be evaluated, together with the matrices $\mathbf{G}_i$, $\mathbf{H}_{ij}$, $\mathbf{K}$ and the vectors $\mathbf{P}_{ij}$. The linear density of the boom has been set as $\rho = .3667kg/m$. The natural frequency of the system resulted to be for the first to modes $\omega_n = .65rad/sec$. The damping matrix $\mathbf{C}$ has been set to be $\mathbf{C} = \text{diag}[.13, .13]10^{-3}N \cdot m \cdot s^2$ whereas the matrix $\mathbf{J}$ comprehensive of the VSCMG point masses inertias has been set to be

$$\mathbf{J} = \begin{pmatrix} 22.9 & 6.4 & 7.6 \\ 6.4 & 128.6 & 5.1 \\ 7.6 & 5.1 & 128.6 \end{pmatrix}.$$

5 Numerical Results

In this section the performance of the controlled system will be tested for a predefined angular acquisition maneuver using different command signals obtained relying upon command shaping techniques. In figure 4 the blue line shows the shape of the reference tracking signal given in terms of the spacecraft angular velocity. The spacecraft starts from a rest condition at a certain attitude and it changes its orientation by performing a rotation around the $[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$ axis (in body reference frame) of an angle of approximately $60°$. The desired tracking signal considered is such that in the ideal case (i.e. rigid dynamics) the angular maneuver is performed in a total time of 104 seconds. The performance of the system in the reference case will be compared with two other maneuvers responding to tracking signals obtained from a command shaping approach. These tracking signals have been designed relying upon a two steps processing technique. First the MATLAB Signal Processing Toolbox has been used in order to design filters capable to remove from the reference tracking signal the natural frequency ω_n of the system. Then the obtained signals have been normalized in order to have the same final value of the attitude parameters. The filters used for this maneuver are shown in figures 2 and 3 respectively.

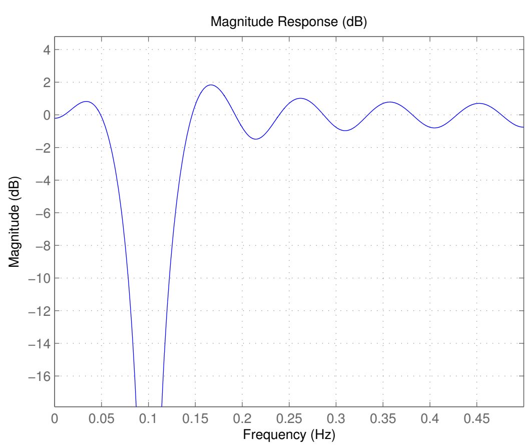

Figure 2: FIR filter designed with the LSF method

The filter in figure 2 is a Finite Impulse Response (FIR) filter designed relying upon a Least Square technique. On the other hand the filter in figure 3 is an Infinite Impulse Response (IIR) elliptic filter. The passband and stopband frequencies for both the two filters are presented in table 1 together with the order of the filters.

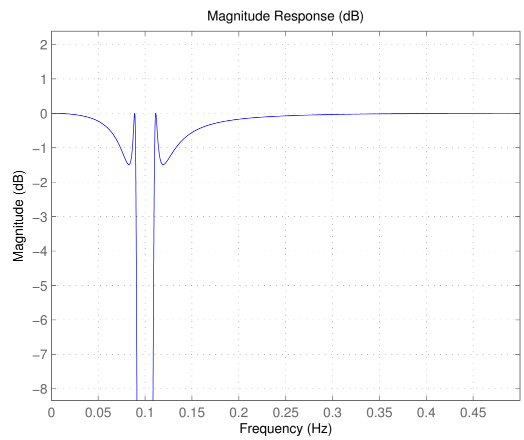

Figure 3: IIR filter designed with the elliptic method

Filter Characteristic	Numerical Value
FIR Passband Up/Low frequency	$\pm 20\% \omega_n$
FIR Stopband Up/Low frequency	$\pm 5\% \omega_n$
FIR Order	20
IIR Passband Up/Low frequency	$\pm 10\% \omega_n$
IIR Stopband Up/Low frequency	$\pm 5\% \omega_n$
IIR Order	4

Table 1: Filters characteristics.

The command signals obtained after filtering are shown in figure 4 (red and green lines). In both the two cases the tracking signal is seen to change significantly with respect to the original one. The IIR filter generates a command signal that requires the satellite to have small oscillations at the end of the principal maneuvre (4 red line).
These oscillation are at the same frequency than the natural one. On the othe hand, the command signal produced

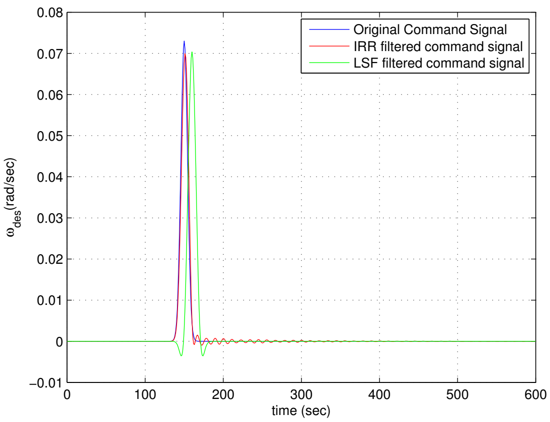

Figure 4: Unfiltered and filtered command signals

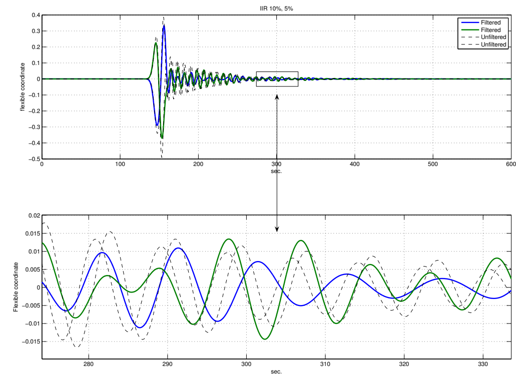

Figure 5: Flexible coordinates for the IIR shaped command signal

by the FIR filter requires the spacecraft first to have a negative angular velocity, then to increase its angular velocity till a peak value (smaller than the one of the original signal) and then decreasing again reaching a negative minimum value (figure 4 green line). Both the two signals have to be then normalized in a way such that the shaped tracking signals yield the same angular acquisition maneuver of the reference case.

The effects of these two command signals on the dynamics of the flexible spacecraft are displayed in figure (5-8) in which the performance of the system in terms of the attitude parameters (MRP) and in terms of induced vibrations are considered.

The first two graphics show the response of the system to the IIR shaped command signal (red line in figure 4). In figure 5 it is shown how the introduction of the shaped command leads to an initial reduction of the system vibration and in a better behaviour of the system in terms of maximum deflection of the flexible appendage.

Nevertheless the amplitude of the oscillations of the system does not decade as fast as the nominal case, therefore, after approximately $300 sec$, the vibration induced by the filtered maneuver are bigger in amplitude with respect to the nominal ones. On the other hand looking at the graph in figure 6 it can be seen that after $250 sec$ the residual vibration on the attitude parameters are not reduced with respect to the case in which the command signal is unfiltered. The command signal obtained after a processing with the IIR filter is then useful when there is not a high precision requirement on the final orientation of the spacecraft. For a given precision of $0.1°$ the use of this shaping filter leads to an improvement of the performance in terms of attitude acquistion time of 14.5% with respect to the unfiltered case. When a more stringent attitude error of $0.04°$ is required the filter is not effective in reducing the acquisition time.

In figure 7 the results given in terms of flexible coordinates for the case in which a LSF shaping filter is considered are shown. In this case the command signal does not require the spacecraft to perform any further oscillation in the angular velocity after the main maneuvre. Both the vibration in the flexible coordinates and in the attitude parameters (see figure 8) are reduced and the maneuver is performed faster with respect to the reference one. If a tolerance of $0.1°$ is imposed on the final orientation of the spacecraft the total maneuver time is reduced of 23.6% with respect to the unfiltered case. When the requirement on the attitude orientation is set to be $0.04°$ the filtered command leads to a decreasing in the attitude acquisition time of 28.2%. Moreover the performance of the system in terms of vibration induced are higher with respect to the ones obtained using an IIR shaping filter. An explaination for the better performances achieved using the LSF-FIR filter can be given taking into account that such filters are designed in order to minimize the difference in energy content between the original and the filtered signals (see

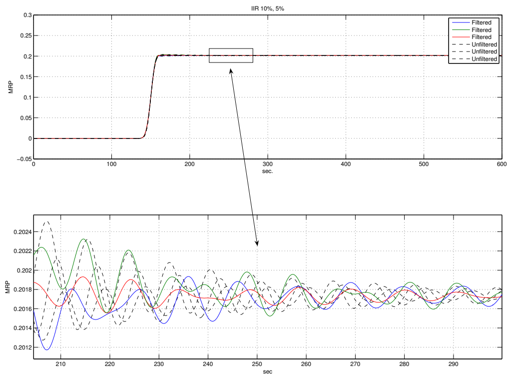

Figure 6: Attitude parameters for the IIR shaped command signal

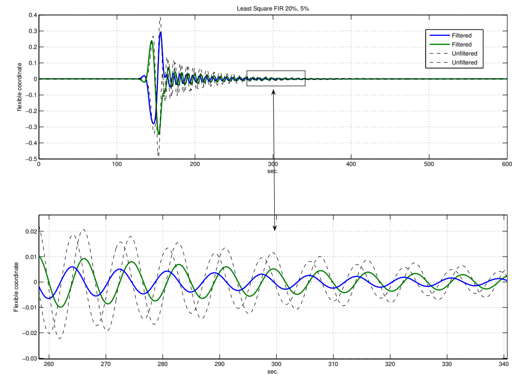

Figure 7: Flexible coordinates for the LSF-FIR shaped command signal

for example [4]). Therefore these kind of filters are best suited when it is necessary to reduce as much as possible the energy of a given signal in a certain frequency bandwidth.

6 Conclusion

In this paper a command shaping techniques has been used in order to increase the performances of a flexible spacecraft equipped with a cluster of Variable Speed Control Moment Gyros performing attitude reorentation maneuvers. For this reason the dynamical model of such a multi-flexible body has been introduced together with a control steering law developed for a rigid spacecraft. The performance of such a system is tested for a three axis angular acquisition maneuver of approximately $60°$. The original tracking signal has been filtered in order to notch out the natural frequency using two different filters. In both the two cases the pre-processing of the tracking signal yields an improvement of the system performance. The IIR filter has shown to decrese the total acquisition time when not a fine performance is required on the final spacecraft orientation. The Least Squares FIR filter has shown to improve the attitude acquisition time of the maneuver also when a more stringent requirement on the final spacecraft orientation is considered.

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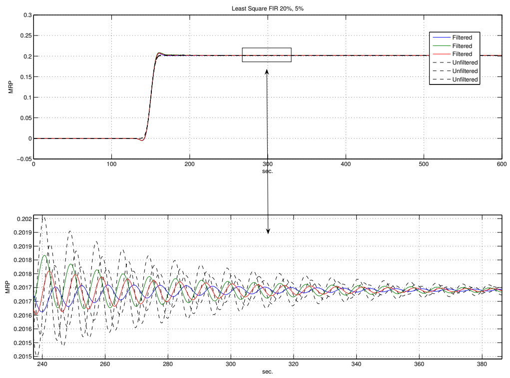

Figure 8: Attitude parameters for the LSF-FIR shaped command signal

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Appendix

In this section a table containing all the symbols appearing in Eq.s(1-4) is presented.

$\mathbf{f}$:	External force acting on the satellite
$\mathbf{a}$:	Acceleration of the spacecraft center of mass in the undeformed configuration
M:	Spacecraft total mass
$\mathbf{L}_i$:	Translational participation factor of the $i-th$ mode $\boldsymbol{\phi}_i$ projected in the $\mathcal{F}_b$ frame.
ε_i:	Flexible coordinate relative to the $i-$th mode.
$\boldsymbol{\omega}$:	Components of the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$ in the $\mathcal{F}_b$ frame. This is the quantity we would like to control, being related to the attitude of the satellite.
$\mathbf{J}$:	Combined inertia matrix of the satellite platform and the point masses m_j placed in $\oplus_{\mathcal{W}_j}$. The frame to which this matrix is referred is $\mathcal{F}_b$ centered in $\oplus_T$.
$\mathbf{C}_j$:	By definition equals $\mathcal{F}_b\mathcal{F}_j^T$ and is therefore a matrix in which the columns represent the components of gimbal, spin and transverse axis of the j-th VSCMG in the $\mathcal{F}_b$ frame. In a VSCMG system this matrix is time varying, being its components related to the gimbal angles.
$\mathbf{Y}_j$:	Inertia matrix of the j-th gimbal-plus-wheel-structure in the $\mathcal{F}_j$ frame centered in $\oplus_{\mathcal{W}_j}$.
$\mathbf{J}_T$:	$\mathbf{J} + \sum \mathbf{C}_j\mathbf{Y}_j\mathbf{C}_j^T + \sum \mathbf{G}_i\varepsilon_i + \sum_i\sum_k \mathbf{H}_{ik}\varepsilon_i\varepsilon_k$ Total inertia matrix of the system (body axis), neglecting the second order terms in ε.
$I^w_{s_j}$:	Moment of inertia around the spin axis of the j-th wheel.
Y_{g_j}:	Moment of inertia around the gimbal axis of the j-th gimbal-plus-wheel-structure.
γ_j:	Gimbal angle of the j-th VSCMG. This is the angle of rotation of the gimbal with respect to its initial position.
Ω_j:	Spin velocity of the j-th wheel.
$\mathbf{s}_j, \mathbf{g}_j, \mathbf{t}_j$:	Components in the $\mathcal{F}_b$ frame of the vectors $\hat{\mathbf{a}}_s, \hat{\mathbf{a}}_g, \hat{\mathbf{a}}_t$.
$\mathbf{S}_i$:	This is the rotational participation factor of the $i-th$ mode projected in the $\mathcal{F}_b$ frame.

$\mathbf{G}_i$:	$\int_{\mathcal{R}} \rho(2(\boldsymbol{\phi}_i \cdot \mathbf{r}^r)\vec{\mathbf{1}} - \boldsymbol{\phi}_i \mathbf{r}^r - \mathbf{r}^r \boldsymbol{\phi}_i) dV$ where $\mathbf{r}^r$ represents the position of a material point mass in the flexible appendage for the undeformed configuration.
$\mathbf{P}_{ik}$:	$\int_{\mathcal{R}} \rho(\boldsymbol{\phi}_i \times \boldsymbol{\phi}_k) dV$
$\mathbf{H}_{ik}$:	$\int_{\mathcal{R}} \rho((\boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_k) - \boldsymbol{\phi}_i \boldsymbol{\phi}_k) dV$
$\mathbf{g}$:	Components of the sum of all external torques acting on the system in the $\mathcal{F}_b$ frame.
ρ	Flexible appendage linear density