(Preprint) AAS 10-171

# DESIGNING ROBUST LOW-THRUST INTERPLANETARY TRAJECTORIES SUBJECT TO ONE TEMPORARY ENGINE FAILURE

**Joris T. Olympio***

The problem of designing low-thrust interplanetary transfer trajectories subject to momentary engine failure is considered. A stochastic optimization problem is posed using chance constraints, and a random engine break-down is simulated. A stochastic approximation method is used to solve the problem. For the mass maximization problem, the switching dates of the bang-bang control are perturbed to respect chance constraints. The choice of the control is thus made on the likeliness that prescribed perturbations of the thrust still allows the mission to be completed. A low-thrust interplanetary transfer problem with one temporary break-down illustrates the methodology.

## INTRODUCTION

The problem of designing low-thrust interplanetary transfer trajectories subject to maximum one random temporary engine failure is considered. As the technology matures, the use of low-thrust propulsion for solar system exploration missions cannot be ignored. Many more complex missions can be performed, although a very long operating time (total thrust duration) is often necessary. In such circumstances, even a mature technology can suffer degradations and failures. With the recent experiences on low-thrust interplanetary missions (Deep Space One, Smart-One, Hayabusa), mission analysts concluded that thrust margins must be included in the design phase of the trajectory to account for trajectory corrections and engine degradations. Despite low-thrust propulsion allowing more flexibility, re-design on-the-fly may not always be possible to satisfy mission objectives. New design methodologies should then be proposed for low-thrust trajectory design.

Typical causes of thrust degradation can be the engine itself or the power system. In particular, for ion propulsion systems the grid can become eroded, resulting in weak ion extraction. At the same time, solar arrays get naturally degraded because of radiation. These cases are usually handled with system margins because basically the engine is still performing but with a degraded performance.

For instance, the NASA/JPL Dawn mission includes system and mission margins to account for a possible under performance of the engine or missed thrust. The mission allows a missed thrust of 28 days at any time. The entire process for increasing the thrust margin to the requirements consists of trading system margins. Additionally, coast arcs are forced for sensitive operations.[1,2]

The future ESA BepiColombo mission includes coast arcs before planetary swing-bys to make sure that the swing-by maneuvers, requiring good accuracy, will be performed as designed. Thus the trajectory has been designed to include 30-day coast arcs before each swing-by and 7-days coast arcs after each swing-by, concurrently with duty cycle reduction accounting for thrust outages.[3]

*Research Fellow, European Space Agency, Advanced Concepts Team, Keplerlaan 1, 2200 AG Noordwijk, The Netherlands. AIAA member.

BepiColombo will also have redundant ion thrusters as degradation is actually expected after a given time.

On November 2009, the JAXA mission Hayabusa suffered an ion engine anomaly. Owing to degradation, the main ion engine had reportedly stopped thrusting during its cruise back to Earth. Currently, among the 4 Hayabusa thrusters, 2 are not being operated because of instability due to degradation, while the recent event may make a third non operational. A technical solution was found, a few days after the anomaly, to maintain the scheduled return cruise thanks to the redundancy of some components. Even though the operational team managed to get back into operational mode, the trajectory necessarily changes, and had the thrusting phase been very sensitive a different scenario may have been required*.

In the aerospace field, examples of engine failure have already been studied for planning back-up trajectories,[4] but it has hardly been considered for low-thrust trajectories of spacecraft. Considering the difficulty of re-planning in the case of low-thrust missions and the low maneuverability of spacecraft dynamics, a low-thrust mission scenario should be designed to account for possible unexpected engine failures. Engine failure can be particularly critical for possible future low-thrust human missions, or low-thrust Near Earth Object deflection missions. Note that the notion of engine failure must be extended to any problems that would require additional low-thrust correction maneuvers (e.g unexpected missed-thrust).

There are two approach to tackle the engine degradation problem: either redundancy of subsystems, or robustness of the control. The study concentrates on the former point. Robustness of the control with respect to engine imperfections plays an important role in the design process. Both Dawn and BepiColombo design methodologies make us wonder if there is an optimal way of taking into account defects in thrust duration, while disregarding the experience of mission analysts. Given that the thrust level imperfection problem can be tackled from a system point of view using thrust-level security factors, or the design methodology of BepiColombo or Dawn, the problem studied in this paper focuses mainly on one momentary engine break-down. Hopefully, we assume that a return-to-operation solution is found during the break-down time to save the mission.

The present article gives a framework for tackling this robust control problem. A model of the perturbed dynamics with a stochastic break-down event is proposed. Using calculus of variations, we parameterize the control to reduce the dimension of the search space. The actual resolution of the problem is done with a modified stochastic approximation process, applied to the problem considered. To illustrate the solution method, we study the case of a low-thrust interplanetary transfer of a spacecraft with momentary engine break-down. We are searching for the control, which allows the target to be reached within a given probability. Lastly, we propose a relationship between the missed-thrust margin function and the probability level of a continuous thrust problem.

## ROBUST STOCHASTIC CONTROL PROBLEM

### Problem Dynamics

In this study, continuous thrust problems are considered. Let us define $\tau$ as the random variable "date of engine break-down" inside a time frame $[t_0, t_f]$, according to a given probability law. The momentary engine break-down duration is given by $\Delta t$. The break-down duration $\Delta t$ can be fixed or random. In this study, however, $\Delta t$ is a constant.

*Press Releases: Asteroid Explorer "HAYABUSA" Ion Engine Anomaly, Nov. 9, 2009, `http://www.jaxa.jp/press/2009/11/20091109_hayabusa_e.html` (retrieved 25 Jan 2010)

We shall consider dynamics of the form:

$$\mathbf{f}(\mathbf{x},\mathbf{u},\delta;t) = \mathbf{f_0}(\mathbf{x};t) + \mathbf{g_1}(\mathbf{x};t)\mathbf{u}(t)\delta(t) + \mathbf{g_2}(\mathbf{x};t)\delta(t) \tag{1}$$

where $\mathbf{f_0}$, $\mathbf{g_1}$ and $\mathbf{g_2}$ are at least continuously differentiable, and the piecewise continuous control is defined by the direction $\mathbf{u}(t)$ and the throttle $\delta(t)$. The control should allow the spacecraft to satisfy some terminal constraints $\psi$ of the final state $\mathbf{x}(t_f)$ and minimize a given objective $J$.

Accounting for the break-down, the dynamics yield:

$$\mathbf{f}_\tau(\mathbf{x},\mathbf{u},\delta;t) = \mathbf{f}(\mathbf{x},\mathbf{u},\delta;t) + [\mathbf{g_1}(\mathbf{x};t)\mathbf{u}(t)\delta(t) + \mathbf{g_2}(\mathbf{x};t)\delta(t)](\xi_\tau(t) - 1) \tag{2}$$

and $\xi_\tau(t)$ can be formulated as:

$$\xi_\tau\left(t\right) = \begin{cases} 0 & \text{on } t \in [\tau, \tau + \Delta t] \\ 1 & \text{elsewhere} \end{cases} \tag{3}$$

The dynamics is seen to be the sum of a deterministic term and a stochastic term. Thus, at some final time $t_f$, the observation of the terminal state $\mathbf{x}(t_f)$ is seen to be the addition of a nominal non-perturbed terminal state, and a bounded perturbation.

Depending on the problem and the control, it may be more convenient to simply formulate the problem with an NLP variable $\tau$ instead of a function $\xi_\tau(t)$.

After a break-down time $\tau$ the problem changes, and the initial control $\mathbf{u}$ cannot be used anymore. A new control $\mathbf{v}_\tau$ is then computed as a feedback of the state $\mathbf{x}(\tau + \Delta t)$ to satisfy the constraint $\psi$. The control $\mathbf{v}_\tau$ may however not exist, but can then be selected as the one that minimizes the constraints.

For many dynamical problems, the break-down influences the controllability of the system. Depending on the problem one must be sure that the system has sufficient time to recover from the break-down, otherwise changing the time $t_f$ may be necessary. In general, re-designing a trajectory on-the-fly does not penalize too much the mass budget thanks to the efficiency of low-thrust propulsion.

**Robust Problem Formulation**

The problem is described as a two-stage stochastic optimization problem[5].[6] The robust problem can however be formulated as an intrinsic first-stage problem:

$$\begin{cases} \min\limits_{\mathbf{u}\in U} & J_0(\mathbf{u}) + \mathbb{E}[J_\tau(\mathbf{u})] \\ s.t. & \\ & \mathbb{P}\left(\psi\left(\mathbf{x}(t_f)\right) = 0\right) \geq p \end{cases} \tag{4}$$

where for instance:

$$J_\tau(\mathbf{u}) = \min_{\mathbf{v}_\tau \in U} j_\tau(\mathbf{u}, \mathbf{v}_\tau) \tag{5}$$

The problem is the minimization of an objective function $J_0$ and the expectation of the cost $J_\tau$, subject to joint probabilistic constraints, with $p \in [\underline{p}, \bar{p}] \subset [0, 1]$. Probabilistic constraints are often called chance constraints, and Prekopa presented a general framework in Refs.[7,8]

The objective function and the constraints are somewhat complex as they handle the changes of the mission when a break-down occurs. For instance, $J_0$ accounts for the first stage of the problem, and $\mathbb{E}[J_\tau(\mathbf{u})]$ accounts for the decision that has to be made after the break-down. For example, in a mass maximization problem, $J_0$ would simply be the mass of the spacecraft at $t = \tau$, while $j_\tau$ would be the cost of the maneuver to recover and satisfy the constraints, when the feedback law (recourse) $\mathbf{v}_\tau$ is used. It is clear though, that in practice $J_\tau$ is only secondary to the actual satisfaction of the constraints and what we are searching is a robust solution $\mathbf{u}^\sharp$.

Because of the dynamics of the problem, both $J_0$ and $J_\tau$ are bounded and continuous. All quantities are thus well defined. In this formulation, the second stage might not have a solution. In addition, the proof of the existence of a solution $\mathbf{u}^\sharp$ to the robust problem is not easy to demonstrate mathematically, and it will not be considered in the current study. Clearly, if a solution cannot be found, this is because the desired mission success rate (probability level) $p$ is too high, or too low, given the current problem assumptions. The lowest acceptable value $\underline{p}$ depends on the probability law of $\tau$ and can be found easily. The highest acceptable value $\bar{p}$, though, requires good knowledge of the problem and experimentation.

The usual approach to defining a probability constraint is to reformulate it as an expectation using an indicator function $\chi$ of a subset. For instance, let us define:

$$\mathbb{P}\left(\psi\left(\mathbf{x}(t_f)\right) = 0\right) = \mathbb{E}\left(\chi_0\left(\psi\left(\mathbf{x}(t_f)\right)\right)\right) \tag{6}$$

where:

$$\chi_A\left(x\right) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

In practice, $A$ accounts for a closed ball $B_\varepsilon[0]$ defined by the numerical tolerance $\varepsilon$ on the constraints.

Taking into account the dynamical equations and the objective function, and introducing the dynamical vector $\boldsymbol{\lambda}(t)$, the overall Lagrangian can thus be written as an expectation:

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}, \boldsymbol{\nu}) = \mathbb{E}\left(L(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}, \boldsymbol{\nu}, \tau)\right) \tag{8}$$

where the loss function is:

$$L(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}, \boldsymbol{\nu}, \tau) = J_0(\mathbf{u}) + J_\tau(\mathbf{u}) + \boldsymbol{\nu}^T\left(\chi_0\left(\psi\left(\mathbf{x}(t_f)\right)\right) - p\right) + \int_{t_0}^{t_f} \boldsymbol{\lambda}^T \left(\frac{d\mathbf{x}}{dt} - f_\tau(\mathbf{x}, \mathbf{u}; t)\right) dt \tag{9}$$

It is important to note that the expectation can seldom be written in a closed form. This makes the resolution of the problem difficult. In general, the overall problem is not convex, and the existence of a saddle point is not guaranteed. There are some methods to improve the local convexity, such as using the augmented Lagrangian. The augmented Lagrangian provides a regularization of the Lagrangian to limit any duality gap and to provide a saddle point in the problem.

In order to solve the problem using a local optimization method (e.g. gradient-based) the Lagrangian must be sufficiently regular. A solution consists in using continuous smoothing techniques. For instance, the $\rho$-Dirac delta function can be used in a convolution to smooth the discontinuity, with a continuation parameter $\rho$, which eventually makes the approximation converge towards the desired function. We take for example:

$$H(x,\epsilon) = \frac{1}{1 + exp\left(-\frac{x}{\epsilon}\right)} \tag{10}$$

$$\tilde{\chi}_D = H(x+D,\epsilon) - H(x-D,\epsilon) \tag{11}$$

This can be visualized in Figure 1. A generalization of the approach to handle discontinuities can be found in Ref.[9]

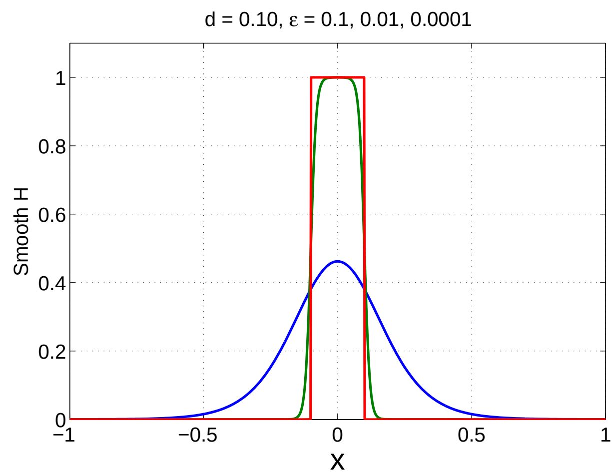


**Figure 1. Example of smoothed indicator function $\chi_A$.**

## Measure of Robustness

So far in this study, the robustness has been connected with the measure of the chance constraints. The satisfaction of the chance constraints to a probability level $p$ and for a given break-down duration $\Delta t$ indicates that the engine can stop firing during $\Delta t$ at any moment, while the mission might still be successful.

A similar metric is given by the missed-thrust margin function.[1] The missed-thrust margin is computed at each date along a trajectory, as the maximum duration the spacecraft can stop thrusting momentarily while still being able to satisfy scientific objectives (e.g. terminal constraints). This metric can be applied to any continuous thrust problem. So far, in the design phase, missed-thrust margins are modified by changing the system parameters such as date, time of flight or thrust amplitude.[1] In the current study, the system is not taken into account per se, and thus the robust trajectory is sought only by modifying the control.

Practically, thrust margins are computed at each date $t$ along the trajectory. During thrust phases, the state $\mathbf{x}(t)$ is propagated ballistically during $\Delta T(t)$ and a new mass maximization optimal control problem is solved from the new state $\mathbf{x}(t + \Delta T(t))$. If the solver converges, the duration $\Delta T(t)$ of the coast arc is increased, and the process restarts. The process is similar for coast phases, although it can be simplified because, by definition, in this case the thrust margin decreases linearly to the thrust margin level of the next thrust point. To find the length of the coast arc, we can follow a dichotomy approach. The maximum coast arc length is then the thrust margin value $\Delta T(t)$. An illustration of the missed-thrust margin is given on Figure 2.

Under the assumptions of this study, we can relate explicitly the missed-thrust margin and a conditional probability measure. For instance, denoting $\Delta T$ the scalar missed-thrust margin function, we have:

$$\begin{aligned} \exists t_A \in [t_0, t_f], \exists \epsilon > 0 \quad & s.t. \quad \forall t \in A = [t_A, t_A + \epsilon], \Delta T(t) \geq \Delta t, \\ & \text{and} \quad \mathbb{P}(\psi = 0 | \tau \in A) = 1 \end{aligned} \tag{12}$$

$$\begin{aligned} \exists t_B \in [t_0, t_f], \exists \epsilon > 0 \quad & s.t. \quad \forall t \in B = [t_B, t_B + \epsilon], \Delta T(t) < \Delta t, \\ & \text{and} \quad \mathbb{P}(\psi = 0 | \tau \in B) = 0 \end{aligned} \tag{13}$$

The conditional probability can be expressed with a composition of the Heaviside step function $H$ and the thrust margin function $\Delta T$. The conditional probability distribution is however not easily written. Then, since we restrict to the thrust intervals, and the characteristic function of the thrust intervals is indeed the thrust throttle function $\delta$, summing over $[t_0, t_f]$ yields:

$$\frac{\int_{t_0}^{t_f} H(\Delta T(x) - \Delta t)\delta(x)dx}{\int_{t_0}^{t_f} \delta(x)dx} = p' \tag{14}$$

with $p' = \mathbb{P}(\psi = 0 | \tau \in T_{thrust})$. On the left hand side, the denominator is indeed the total thrust time, while the numerator represents the cumulative measure of the periods during which the missed-thrust margin is greater than the break-down duration $\Delta t$, as illustrated in Figure 2.

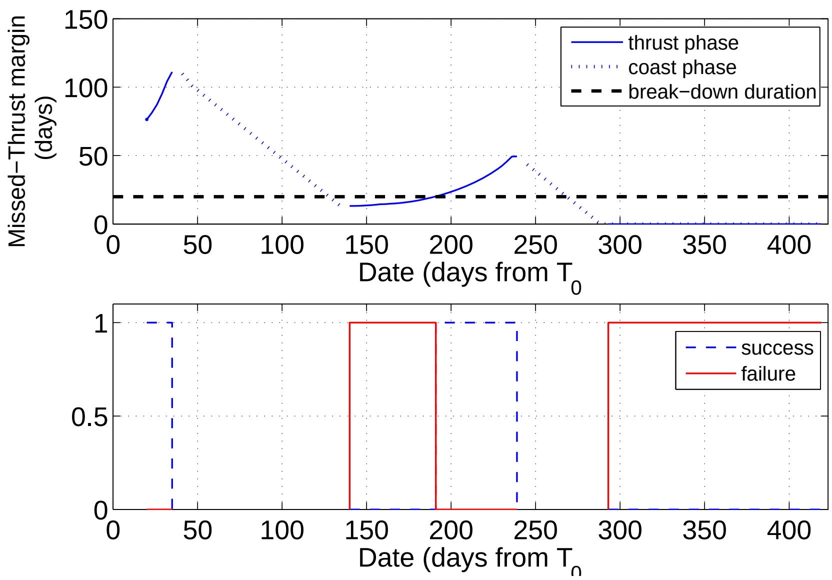


**Figure 2. Missed-thrust margin function and probabilistic interpretation.**

It is interesting to see that the stochastic analysis can be confirmed by a deterministic formula, and this will be used when assessing the results of the problem we propose to solve.

Note at this point that even though in this study the solution method is about solving the stochastic optimization problem, because of the equivalence provide by Equation ( 14) the problem can also be solve using a deterministic approach. Although, solving the deterministic problem would certainly require putting inequality path constraints all along the trajectory, with the difficulty of having a very expensive constraint function because $\Delta T$ cannot easily be written in a closed form. Note also that the missed-thrust margin is not required to exceed $\Delta t$ every where, which would have been the case with path constraints, probably resulting in excessive robustness. We expect that robustness will impact the mass budget. A stochastic problem formulation is relevant because a break-down is an event that should not happen in the ideal case, and hopefully, break-down will not happen in future missions. So, designing a trajectory according to our approach is actually equivalent to the one based on the missed-thrust margin, although our approach does not trade system margins and provides instead a system-independent trajectory design method.

## STOCHASTIC OPTIMIZATION METHODS

Few algorithms are able to solve a stochastic problem. The stochastic problem we are considering is one defined by a stochastic objective function and constraints, as defined by Equation (4). For this problem, neither the objective function nor the constraints are directly available. All we observe is an a-priori fluctuating quantity, which can be averaged over runs. A common idea to solve this type of problem is to perform an optimization by simulation. A Monte-Carlo process provides an estimate of the expectation, while deterministic tools can be used for the minimization of this estimate. It is clear that because the expectation has no algebraic equivalent, its calculation can be very expensive.

### Sample Path Method

A common approach to solve stochastic optimization problems is by formulating a new problem using the sample path method, also known as the Sample Average Approximation method (SAA). The basic idea is indeed to replace the expectation by a finite sum over a fixed set of independent and identically distributed realizations of the random variable $\tau$, and to optimize the new objective function.[10,11,12,13] So:

$$\min_{\mathbf{u}} \frac{1}{N} \sum_{n=1}^{N} J_{\tau}(\mathbf{u})$$

The choice of the random set $\{\tau\}_n$ depends on the reference value for $\mathbf{u}$. As the optimization is then performed over a fixed random set $\{\tau\}_n$, the optimization is indeed deterministic. The resolution of such a problem can thus be very interesting, although it can also be expensive particularly for non linear problems, when the problem structure cannot be simplified to cope with high dimensionality. An interesting point is that this method does not require any parameter tuning. The almost surely convergence requires however some mild conditions.The sample size should theoretically be impractically large to reduce the error made on the solution. Some methods must be implemented to account for the high computational complexity, by working on several problems with increasing sample size with an adaptive error tolerance.

### Stochastic Approximation Methods

An other approach is proposed by Stochastic Approximation algorithms (SA). They include the Robbins-Monro process, the Kiefer-Wolfowitz procedure for minimization, and variants.[14,15,16,17]

The Robbins-Monro process tackles the root-finding problem, denoted as the Stochastic Root-Finding Problem (SRFP) in the literature,

$$\mathbb{E}(h) = \alpha$$

The Kiefer-Wolfowitz procedure solves the minimization problem

$$\min \mathbb{E}(h)$$

The Robbins-Monro process uses a Newton's method formulation, while the Kiefer-Wolfowitz procedure is closer to a stochastic gradient algorithm. The recursive procedure is of the form

$$\mathbf{u}^{k+1} = \mathbf{u}^k - \epsilon_k U(\mathbf{u}^k, \tau)^{-1} G(\mathbf{u}^k, \tau) \tag{15}$$

where $U$ can be the identity matrix (Robbins Monro) or an estimator of the Hessian of $\mathbb{E}[h]$ (Kiefer-Wolfowitz), and $G$ is an estimator (ideally unbiased) of $\mathbb{E}[h]$ (Robbins Monro) or the gradient of $\mathbb{E}[h]$ (Kiefer-Wolfowitz). The estimators are thus constructed on the evaluations of the loss function $h$.

Considering the transformation by Equation (6), a Robbins-Monro process is a perfect candidate for the constraint part, while the Kiefer-Wolfowitz procedure could handle the minimization of the average of the objective function. Note however that the minimization problem can be tackled as a root-finding problem too. There are however few variants of Kiefer-Wolfowitz to handle equality constraints. In fact, it would have been interesting to minimize the Lagrangian, but the difficult proof of existence of a saddle point makes this approach hazardous.

An extension of the Robbins-Monro root-finding procedure to multi-variate functions can be found in the work of Blum.[18] Later, Ruppert and others proposed a less restrictive approach than Blum, by deriving a Newton-Raphson method.[19,20] Our algorithm is based on these later works. We will assume that the set $U$ is connected and convex, or at least that it is locally provided we start from a good initial guess. We are indeed only interested in a local solution.

**Convergence**

The rate of convergence to a solution of the Robbins Monro process is faster than the rate of convergence of the Kiefer Wolfowitz.

The choice of $\epsilon_k$ is critical for a good convergence. In particular, it must tend toward 0 ($\sigma$-suite), and slowly, so as to average out the effect of noise. The most common form is for instance:

$$\epsilon_k = \frac{\alpha}{(\beta + k)^\gamma} \tag{16}$$

where $\alpha > 0$, $\beta > 1$ and $\gamma > 0$, such that

$$\epsilon_k > 0, \quad \sum_{k \geq 1} \epsilon_k = \infty, \quad \sum_{k \geq 1} \epsilon_k^2 < \infty \tag{17}$$

The choice of the coefficients is based on knowledge of the problem and to some extent on experimentations. In Ref.,[21] the authors provide an analysis on the selection of the step for a robust SA algorithm. The result is that the step must be sufficiently large to ensure a sufficient rate of convergence, but not too large to avoid noisy behavior. We should thus allow large steps for the first iterations. Stochastic approximation algorithms however converge slowly,[22] and it can take an indefinite number of iterations to actually converge. A sufficient number of iteration should thus be given such that an acceptable solution can be found.

The use of random variables allow the formulation of a confidence interval (hypercube), assuming a normal distribution,

$$[\bar{\mathbf{u}} - t\sigma_{FD}, \bar{\mathbf{u}} + t\sigma_{FD}] \ni [\mathbf{u}_k - \epsilon; \mathbf{u}_k + \epsilon] \tag{18}$$

such that: $P(-t \leq u_i \leq t) = c$ with $c$ the confidence level, $\sigma$ stands for the standard deviation and $\bar{u}$ the mean.[17] Ideally, convergence to a given tolerance $\epsilon$ is achieved when the current value lies in this hypercube.

## SOLUTION METHOD

### Robust Problem Setup

Denote respectively the disjoint close intervals $T^{\bar{u}}_{coast}$ and $T^{\bar{u}}_{thrust}$, contained in $[t_0, t_f]$, during which the spacecraft is coasting, respectively thrusting, for the nominal solution $\bar{u}$. $T^{\bar{u}}_{coast}$ and $T^{\bar{u}}_{thrust}$ can be union of sub-intervals depending on the control structure,

$$T^{\bar{u}}_{thrust} = \cup_i T^{\bar{u},i}_{thrust} \quad \text{and} \quad T^{\bar{u}}_{coast} \cap T^{\bar{u}}_{thrust} = \varnothing$$

The random variable $\tau$ follows the probability law:

$$\mathbb{P}(\tau \in T^{\bar{u}}_{thrust}) = 1 - q \tag{19}$$

to simulate the break-down event, where $q \in ]0, 1[$. The random variable is uniformly distributed between the sub-intervals of $T^{\bar{u}}_{thrust}$. We will abusively refer to the value $1 - q$ as the failure rate of the engine, although it is understood that it is not a failure rate per unit time but mainly the probability that the engine fails during the mission. The deterministic case is given by $q = 1$ and methods of solution are well known so this case will not be considered for the current study. Similarly, $q = 0$ makes the problem ill-posed and implies that there will always be an engine failure, which is also unrealistic considering current technology.

The computation of the probabilities can be simplified. First, note that for any event $A$, the probability $\mathbb{P}(\psi(\mathbf{x}(t_f)) = 0|A)$ can be equivalent to $\prod_i \mathbb{P}(\psi_i(\mathbf{x}(t_f)) = 0|A)$. We should pose:

$$\mathbb{P}(\psi(\mathbf{x}(t_f)) = 0|\tau \in T^{\bar{u}}_{coast}) = 1 \tag{20}$$

which implies that only a break-down of the engine can cause a violation of the terminal constraints.

In addition, using conditional probability, we have simply:

$$\mathbb{P}(\psi(\mathbf{x}(t_f)) = 0) = q + (1 - q)\mathbb{P}\left(\psi(\mathbf{x}(t_f)) = 0|\tau \in T^{\bar{u}}_{thrust}\right) \tag{21}$$

This means that the chance constraint only depends on the conditional chance constraint when a break-down occurs during thrust arcs. This obviously shows, given the probability law for $\tau$, the necessary condition $\underline{p} = q$. Equation (21) also simplifies the numerical complexity and shows that it is not necessary to run the algorithm when the random variable $\tau \in T^{\bar{u}}_{coast}$.

For convenience we should thus set

$$\mathbb{P}\left(\psi(\mathbf{x}(t_f)) = 0|\tau \in T^{\bar{u}}_{thrust}\right) \geq p' \tag{22}$$

in place of the chance constraints formulated in Equation (4), while ensuring that the equivalent unperturbed problem constraints are always satisfied. In this particular case, the two-stage problem can be simplified to one single-stage problem.

Thus, since $\psi\left(\mathbf{x}(t_f)\right)$ and $\tau$ are not independent random variables, with Equation (6), the following sum is considered:

$$\mathbb{E}\left(\chi_0\left(\psi\left(\mathbf{x}(t_f)\right)\right), \tau \in T^{\bar{u}}_{thrust}\right) = \sum_i \mathbb{P}\left(\tau \in T^{\bar{u},i}_{thrust}\right) \mathbb{E}\left(\chi_0\left(\psi\left(\mathbf{x}(t_f)\right)\right), \tau \in T^{\bar{u},i}_{thrust}\right) \quad (23)$$

The stochastic approximation algorithm can be computed separately for different draws of the variable $\tau$ in $T^{\bar{u},i}_{thrust}$. The advantage is twofold: the process becomes less noisy as there are typically no jumps from one subinterval $T^{\bar{u},i}_{thrust}$ to another, and the analysis is simplified because we can consider separate cases. We can actually identify which thrust arc is the most critical.

Note though, that since the distribution function depends on the current solution $\mathbf{u}$, an unbiased estimator of the expectations cannot be constructed by simply measuring $\psi$, but depends also on the density function of $\tau$.

**Robust Control and Calculus of Variations**

Formulating the objective function in the Mayer form, the Hamiltonian can be defined as:

$$H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}, \delta; t) = \boldsymbol{\lambda}^T \mathbf{f}_\tau(\mathbf{x}, \mathbf{u}, \delta; t) \quad (24)$$

The Euler-Lagrange equations are used to compute $\boldsymbol{\lambda}(t)$ and $\mathbf{x}(t)$ from an initial point $\boldsymbol{\lambda}(t_0) \neq 0$.

Equation (2) and the linearity properties of the expectation operator, give:

$$\min_{\|\mathbf{u}\|=1} \mathbb{E}\left(H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}, \delta; t)\right) \equiv \min_{\|\mathbf{u}\|=1} \mathbb{E}\left(\boldsymbol{\lambda}(t)^T \mathbf{g_1}(\mathbf{x}; t)\mathbf{u}(t)\xi_\tau(t)\right) \quad (25)$$

As $\xi_\tau(t)$ is a positive function, this shows that the robust control direction $\mathbf{u}$ depends only on the co-state vector $\boldsymbol{\lambda}$ in the general case (e.g. when $\xi_\tau(t)$ or $\delta(t)$ equals 0, $\mathbf{u}$ can take any direction). In fact, it shows that the robustness explicitly depends on the control structure and the switching times (e.g. the duty cycle). Also:

$$\min_{0\leq\delta\leq 1} \mathbb{E}\left(H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}, \delta; t)\right) \equiv \min_{0\leq\delta\leq 1} \mathbb{E}\left(S(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}, \xi_\tau; t)\delta(t)\right) \quad (26)$$

$$S(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}, \xi_\tau; t) = \boldsymbol{\lambda}(t)^T \left(\mathbf{g_1}(\mathbf{x}; t)\mathbf{u}(t)\xi_\tau(t) + \mathbf{g_2}(\mathbf{x}; t)\xi_\tau(t)\right) \quad (27)$$

The function $S$ should help defining the control structure that would be used as a bang-bang policy. However, as we cannot easily formulate robustness as an analytic function, calculus of variation and the maximum principle cannot easily provide the switching function for the robust control $[\mathbf{u}^\sharp, \delta^\sharp]$. For the same reason, the boundary value problem is not easy to solve. The limitation essentially comes from the impossibility of computing $\mathbb{E}(\xi_\tau(t))$ analytically, for the general case.

$$\mathbb{E}(\xi_\tau(t)) = \int_{[t_0, t_f]} \xi_\tau(t) dP(\tau) \quad (28)$$

We should actually wonder if the control is still bang-bang as it is for the maximum mass optimization problem. So, although the control direction may explicitly depend on the co-state vector, we shall "neglect" the transversality conditions.

Consequently, as we have free boundary conditions for the co-state vector at terminal time, we have free variables. We shall thus impose, a-priori, the robust control structure $\delta^\sharp$, find the robust switching times and the initial co-state vector, and eventually iterate over different structures. The robust control structure must not only satisfy the chance constraint but also minimize on average the cost function. By the nature of the problem, we will indeed theoretically span the whole manifold of the solution of the Euler-Lagrange equations, starting from the selected initial point and satisfying the terminal state conditions, to find the robust control.

To reduce the sensitivity of the problem, an adjoint control transformation is used.[23] This transformation replaces the co-state vector with a vector with reduced sensitivity and of more physical meaning.

### Modified Stochastic Process

We based our algorithm on the Robbins Monro process. We discarded the SAA procedure, despite its good precision, because of the potential computational burden, and during preliminary work we did not manage to find a solution to the deterministic problem. For all stochastic approximation processes, the basic idea is to construct iteratively an estimation of $\mathbb{E}[.]$, and a sequence $\mathbf{u}_n$, which eventually converges to the optimum $\mathbf{u}^\sharp$. We want to solve

$$\mathbb{E}(\chi_0(\psi)) = p \qquad \text{for} \quad \mathbf{T}, \boldsymbol{\lambda}_0 \tag{29}$$

Note that we drop the objective function to solve only the boundary value problem. The decision vector of the problem considers the switching dates $\mathbf{T}$ and the co-state vector $\boldsymbol{\lambda}_0$. A stochastic step cannot be applied efficiently to the co-state vector $\boldsymbol{\lambda}_0$, because by definition it is a sensitivity vector, and according to Equation (18), it would not be possible to compute it accurately. Our objective is to be relatively close to the optimal trajectory in order not to penalize the objective function. This also allows the initial co-state vector to be used as an initial guess.

We implement a two stage process. The first stage, that we refer to as a stochastic predictor, is a Robbins-Monro step working on the switching dates only for the chance constraints. The second stage performs a Newton-Raphson step over the complete decision vector to satisfy the state constraint $\psi$ of the unperturbed problem.

Taking Equation (15), the recursive SA procedure provides the stochastic prediction

$$\mathbf{T}^{k+1/2} = \mathbf{T}^k - \epsilon_k G(\mathbf{T}^k, {\boldsymbol{\lambda}_0}^k, \tau)^{-1} F(\mathbf{T}^k, {\boldsymbol{\lambda}_0}^k, \tau) \tag{30}$$

where $G$ must be seen as an estimator of the Jacobian matrix $\nabla_\mathbf{T}\mathbb{E}[\chi_0(\psi)]$, and $F$ is an estimator of $\mathbb{E}[\chi_0(\psi)]$. The estimators are thus constructed on the observations $\chi_0(\psi)$. In fact, $F$ does not need to be an estimator constructed from the $N$ past iterations, because in the Robbins-Monro process, $F$ is averaged across iterations.[17] Based on the work of Kiefer-Wolfowitz, $G$ can be computed as a stochastic finite-difference, and the order of magnitude of the bias is the square of the length of a finite-difference step.

In Spall,[20] a function is minimized and an approximate Hessian is averaged. This method is reminiscent of a Quasi-Newton minimization methods, where the Hessian is updated according to previous measures of the gradient. Because the problem considered here is connected with finding a zero, the Hessian cannot be used. Nonetheless, we can approximate the gradient $G$ as:

$$G_{k+1} = \frac{G_k + \epsilon_k(k+1)G(\mathbf{T}^k, {\boldsymbol{\lambda}_0}^k, \tau)}{k} \tag{31}$$

This has the advantage of not computing directly the gradient of the estimate of the expectation, which can be expensive. Matrix $G_k$ is inverted using a Moore-Penrose pseudo-inverse operation.

The correction step is performed by computing

$$\begin{bmatrix} \mathbf{T}^{k+1} \\ \boldsymbol{\lambda}_0{}^{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{T}^{k+1/2} \\ \boldsymbol{\lambda}_0{}^{k} \end{bmatrix} - g(\mathbf{T}^{k+1/2}, \boldsymbol{\lambda}_0{}^{k})^{-1} \psi(\mathbf{T}^{k+1/2}, \boldsymbol{\lambda}_0{}^{k}) \tag{32}$$

where $g$ is the Jacobian matrix $\nabla_{\mathbf{T},\boldsymbol{\lambda}_0} \psi$. For the correction step, the perturbation is not taken into account.

This process is repeated for a given number of iterations. Ideally, the step $\epsilon_k$ should not be selected too small (e.g. larger than a Kantorovich-like step) for the process to explore the state space.

A thorough investigation of the property of this algorithm and its asymptotic behavior is beyond the scope of this article. However, for sufficiently large $k$ the stochastic steps become negligible, while the correction stage performs the necessary steps such that the final unperturbed solution is valid, at least for the deterministic problem. When the stochastic step is dominant, the algorithm is expected to have the same behavior as the Robbins-Monro process.

Since the SA process is closed to the simplest Newton method and the problem is not convex, many initial guesses may be required before convergence.

**Robust Problem Initial Guess**

The generation of an initial guess addresses the choice of thrust durations and of the co-state vector $\boldsymbol{\lambda}(t_0)$. We assume the final arc is thrusting and that it is not at all robust

$$\mathbb{P}\left(\psi\left(\mathbf{x}(t_f)\right) = 0 | \tau \in T_{thrust}^{\bar{u}, n_{thrust}}\right) = 0 \qquad \text{wp } 1$$

with $n_{thrust}$ the number of thrust arcs. Assume also that all preceding thrust arcs are fully robust,

$$\mathbb{P}\left(\psi\left(\mathbf{x}(t_f)\right) = 0 | \tau \in T_{thrust}^{\bar{u}, 1..n_{thrust}-1}\right) = 1 \qquad \text{wp } 1$$

Under this special case of choice of control structure and probability distribution, the first thrust arcs should increase as the probability level $p$ increases to reduce as much as probabilistically necessary the final thrust arc. Denoting the length of an interval $\mathbb{I}$ by $M(\mathbb{I})$, we have the following:

$$\sum_{i=1}^{n_{thrust}-1} M(T_{thrust}^{i}) = p' \sum_{i=1}^{n_{thrust}} M(T_{thrust}^{i}) \tag{33}$$

Using Equation (21) and Equation (23), this defines a new fully deterministic problem that can be easily solved. A non-linear solver can be used to find how to share the thrust-time resource between the thrust arcs and provide a valid $\boldsymbol{\lambda}(t_0)$. Note however, that this initial guess does not take into account the break-down duration, and it is the solution for the robust problem when $\Delta t = 0$, or for sufficiently small values of $\Delta t$.

It is very likely that the problems we are considering have many solutions, and thus are not convex. However, we can expect this initial guess to place the initial point in a convex neighborhood such that our SA process converges, if a solution exists in that neighborhood.

## EXAMPLE: LOW-THRUST INTERPLANETARY TRANSFER

### Problem description

The Low-Thrust interplanetary transfer considered is a transfer from Earth to Mars. The two-body dynamical equations of the problem are:

$$f(\mathbf{x}, \mathbf{u}, \delta; t) = \frac{d}{dt}\begin{bmatrix}\mathbf{r}\\ \mathbf{v}\\ m\end{bmatrix} = \begin{bmatrix}\mathbf{v}\\ -\frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r}\\ 0\end{bmatrix} + \begin{bmatrix}0\\ \frac{F}{m}\\ 0\end{bmatrix}\mathbf{u}\delta + \begin{bmatrix}0\\ 0\\ -q\end{bmatrix}\delta \tag{34}$$

where $\mathbf{r}$ and $\mathbf{v}$ are the spacecraft inertial position and velocity. $m$ is the spacecraft mass, $F$ the thrust force amplitude, $q$ the mass flow rate when firing and $\mu$ is the central body gravitational constant.

The control is defined by the direction $\mathbf{u}$ and amplitude $\delta$.

$$\|\mathbf{u}\| = 1 \tag{35}$$

$$0 \leq \delta \leq 1 \tag{36}$$

The terminal constraints for the rendezvous maneuver:

$$\psi(\mathbf{x}; t_f) = \begin{bmatrix}\mathbf{r}(t_f) - \mathbf{r}_p(t_f)\\ \mathbf{v}(t_f) - \mathbf{v}_p(t_f)\end{bmatrix} \tag{37}$$

where $\mathbf{r}_p(t_f)$ and $\mathbf{v}_p(t_f)$ are respectively the arrival planet heliocentric position and velocity at date $t_f$. To prevent obvious results, the last arc is forced to be thrusting. This means that the target is reached, or the scientific objective is performed, exactly at date $t_f$.

The objective is to minimize fuel consumption, and thus the objective function is:

$$J(\mathbf{u}, \delta) = -m(t_f) \tag{38}$$

The problem parameters are described in Table 1. For the sake of illustration, a NEP engine (constant-thrust) is considered.

| | | |
|---|---|---|
| $F$ | Thrust Force | $0.294\ N$ |
| $P$ | Power | $4.6\ kW$ |
| $m$ | Initial S/C mass | $1000\ kg$ |
| $Isp$ | Specific Impulse | $3170\ s$ |
| $t_f - t_0$ | Time of Flight | $423\ days$ |

**Table 1. Low-Thrust Transfer problem setup**

### Optimal Control Problem Solution

Using the maximum principle, the optimal direction is given by the primer vector direction $\boldsymbol{\lambda_V}$, while the thrust amplitude is found using a switching function. The unknown in the optimal control problem is the vector $\boldsymbol{\lambda}(t_0)$, and is selected such that transversality conditions and terminal state constraints are satisfied.

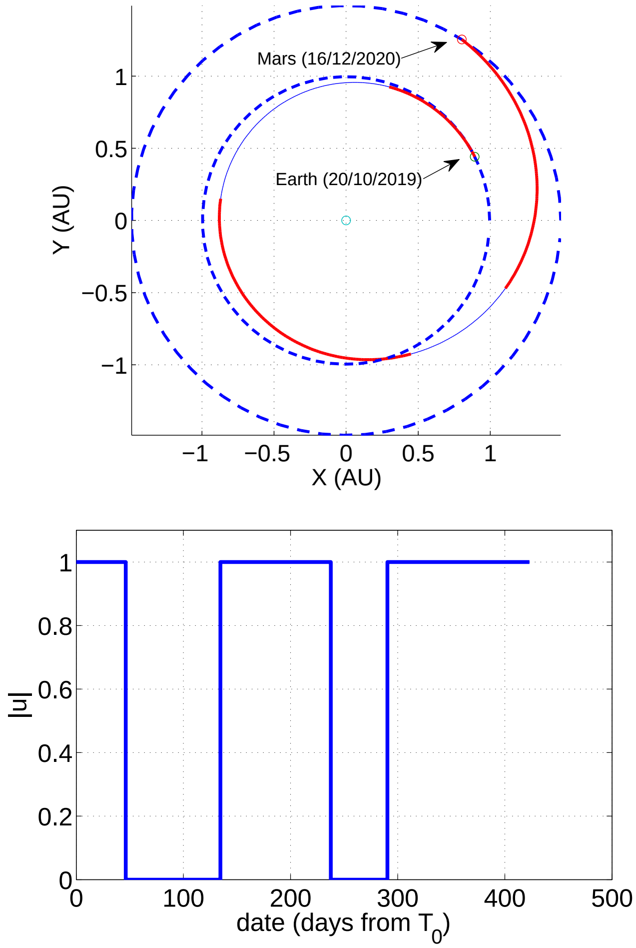


**Figure 3. Optimal Control Solution for the Low-Thrust Transfer problem.**

| break-down duration $\Delta t$ (days) | constraint satisfaction (%) partial $p'$ ($P_1$, $P_2$, $P_3$) | overall $p$ |
|---|---|---|
| 0 | $100\ (100, 100, 100)$ | 100 |
| 5 | $53\ (100, 100, 0)$ | 76.5 |
| 10 | $52.9\ (100, 99, 0)$ | 76.5 |
| 15 | $42.2\ (100, 70.4, 0)$ | 71.1 |
| 20 | $32.4\ (100, 43.8, 0)$ | 66.2 |

**Table 2. Robustness of the Optimal Low-Thrust Transfer solution. Probability of satisfying the constraint for a given break-down duration $\Delta t$ (with reliability $q = 50\%$) computed by a Monte-Carlo simulation (1000 runs). $P_i$ refers to the partial probability when the perturbation appears on thrust arc $i$ only.**

The optimal control solution is readily computed. It is shown in Figure 3. Final mass is $769.6kg$. This trajectory has been selected to provide long thrust arc, regardless of the final mass for a given time frame.

The probabilities to satisfy the constraint are given in Table 2. The table confirms that the final thrust arc is critical, while the first thrust arc is completely robust to the perturbations considered. This is also illustrated on Figure 2. In fact, from the table, if the break-down length is 0, 5 or 10 days, the resolution of the problem resume to designing appropriately the last thrust segment only. This can be done either by introducing a last coast arc, or by reducing the size of the segment until that a break-down occurring on that final thrust arc implies a satisfaction of the overall chance constraint. If the break-down duration is 20 days or more, the second thrust arc must also be re-designed.

### Robust Solution with Stochastic Approximation

The basic idea is to parametrize the burn arcs, or switching dates, and then to look for the control that both satisfies the bang-bang structure and the chance constraint. In this example, the time of flight is kept constant because the optimal solution shows sufficient recovery time thanks to the total duration of the coast arcs.

The chance constraint level is fixed to $p = 90\%$ (or $p' = 80\%$) and we fix the reliability rate to $q = 50\%$ for a better contrast between the different solutions. The break-down duration is fixed at $\Delta t = 20$ days, from Table 2. Coefficients $\alpha$, $\beta$ and $\gamma$ have to be selected according to the control structure.

We shall study 3 different control structures: TC+T, 2 TC+T and 3 TC+C where 'T' denotes a thrust arc and 'C' a coast arc. The 2 TC+T case is the control structure of the optimal solution found previously. The control, however, does not need to finish with a thrust arc. In addition, a break-down on the final thrust does not necessarily lead to a violation of the constraints ($\varepsilon \leq 10^{-6}$).

Table 3 gives a description of the robust control. The robust control solution is depicted in Figure 4, and should be compared with the nominal optimal control of Figure 3. Figure 5 gives the evaluation of an estimator of $\mathbb{E}[\chi(\psi)]$ along the iterations of the algorithm. Because of a good initial guess, the process goes easily toward a valid robust solution although convergence is in general slow. The displayed estimator is constructed as a mean of the observation of $\chi(\psi)$ over the past $N$ iterations. It is only when the solution becomes steady that the estimator is unbiased. The value obtained near the end of the curve is then representative of the chance constraint value. Satisfaction of the chance constraint is achieved to an accuracy $0.1\%$ (to be compared with $p'$).

The total thrust duration increases from 280 days to about 320 days from the optimal to the most robust solution found. This increase represents the necessary trade-off between optimality and robustness, and the cost penalty depends approximately on the time difference. The difference in final mass could be compared to the mass loss during $\Delta t$ (about 16 $kg$). The most robust control we found thus has a penalty of twice this value and can be a quite significant penalty for any mission mass budget. Note though, that the assumption made on the failure rate, $q$, along with the thrust force amplitude, can justify the important penalty. In addition, a few percent higher $p$ increases the cost by a few kilograms. Thus, increasing the robustness cannot result in a reduction in total thrust time (or cost) because the chance that a break-down occurs stays the same.

| control structure | arc durations (days) | total thrust duration (days) | mass $m_f$ (kg) |
|---|---|---|---|
| optimal (TCTCT) | 46.3, 88.3, 103.0, 52.9, 132.6 | 280 | 769.6 |
| robust TCT | 290.2, 60.3, 72.5 | 363 | 703.6 |
| robust 2 TC+T | 96.0, 57.9, 166.1, 38.2, 64.8 | 326.9 | 732.8 |
| robust 3 TC+T | 136.9, 61.4, 79.8, 8.9, 39.9, 31.9, 64.2 | 320 | 737.8 |

**Table 3. Low-Thrust robust transfer problem results,** $p \geq 90\%$

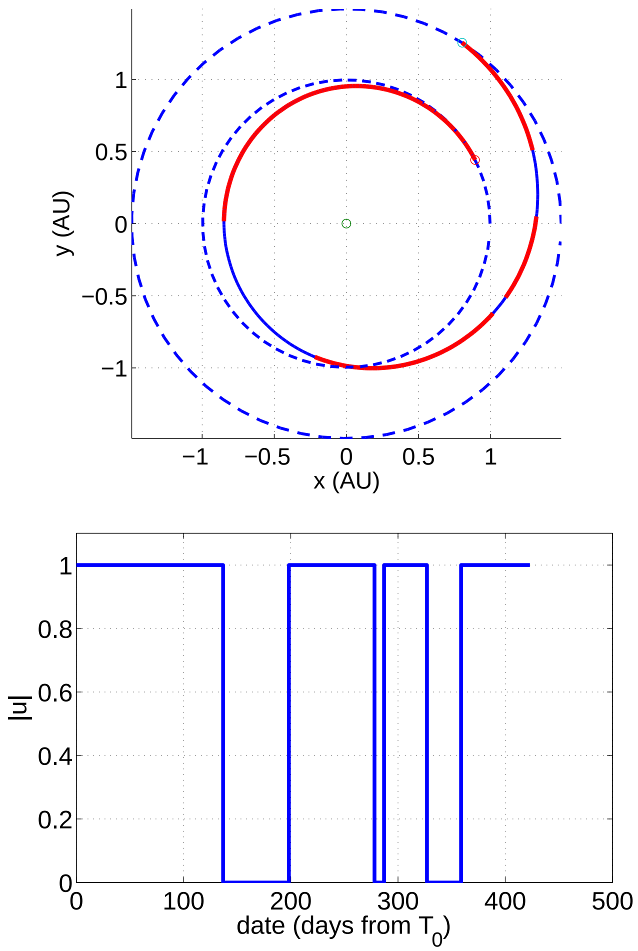


**Figure 4. Robust solution of the Low-Thrust Transfer problem**

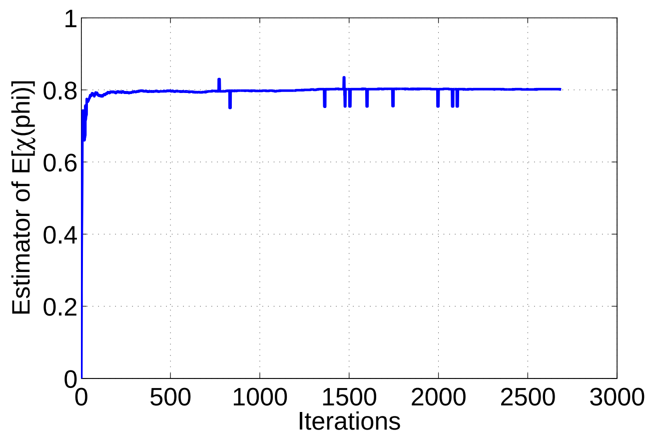


**Figure 5. Iterations of the algorithm for the Low-Thrust Transfer problem. TCTCT case.**

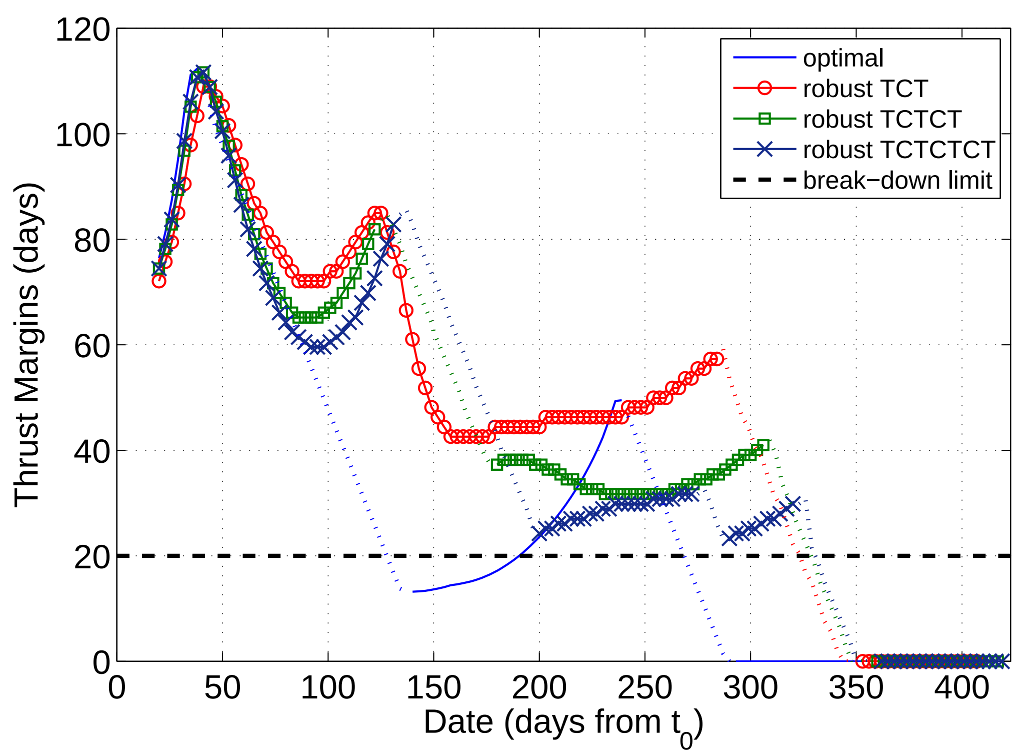


**Figure 6. Missed-thrust margins for the OCP solution and robust solutions according to the control structure. Coast phases are represented in dashed, and thrust phases are solid. Thrust margins are computed with $1$ day accuracy.**

## Missed-Thrust Margins

The robust solution is checked by performing a Monte-Carlo simulation (10000 parallel runs), and we compute concurrently the missed-thrust margin. Results are shown in Figure 6. The missed-thrust margin does not have any meaning, in our problem, during coast arcs. These phases are depicted by dotted lines on the graph.

As proved by the Monte-Carlo simulation in Table 2, the optimal trajectory is not robust enough, and the missed-thrust margin function shows that indeed the missed-thrust gets below the $\Delta t$ level for the second and third thrust arcs. This means that any break-down when the missed-thrust margin gets below the $\Delta t$ level results in a mission failure.

All the robust solutions that we reported have a thrust margin above the $\Delta t$ level except for the last arc. The process has shifted the missed-thrust margins by the necessary amount to satisfy the chance constraint. The last thrust arc cannot have a high missed-thrust margin. By definition, it necessarily reaches $0$ from $t_f - \Delta t$. Had we allowed a final coast arc, the missed-thrust would have been strictly positive, and its value would be at least the duration of the last imposed coast arc. Since for operation, a final coast arc is usually imposed, our approach is indeed very conservative. Noticeably though, all robust solutions have a short final thrust arc compared to the optimal solution.

The shape of the missed-thrust shows that there exists very sensible and critical thrust arcs along the trajectory, besides the final thrust arc, which require specific attention. The missed-thrust margins of the first thrust arc all show a lower part that may well go below the $\Delta t$ limit.

## CONCLUSION

The design of a robust continuous thrust trajectory subject to momentary engine break-down is a difficult problem. With mild assumptions, the paper presents a simple possible approach to solve the problem. The problem is formulated as a stochastic optimization problem, where constraints are formulated as probability constraints. A solution is sought to satisfy the constraint with a given probability, by estimating the expectation of the stochastic Lagrangian and using a stochastic approximation process.

The example of low-thrust interplanetary transfer has been studied. The algorithm shows an efficient way of designing robust trajectories.

Missed-thrust margins have been used to assess the robustness of the trajectory. Also, the system parameters are not changed to increase the robustness as it is done with margin approaches. We eventually proposed a relationship between the missed-thrust margin function and a conditional probability measure.

Essentially the method allows to tackle the problem of missed-thrust margin design while not needing an explicit formulation of the missed-thrust margin or an analytical expression of robust criteria.

Future work shall handle the case of the SEP engine, different formulations of failure such as the use of Mean Time Between Failure or Failure In Time. It shall also include a deterministic approach, and the case of multiple break-downs.

## REFERENCES

[1] M. Rayman and S. Williams, “Design of the first interplanetary Solar Electric Propulsion Mission,” *Journal of Spacecraft and Rocket*, No. 39, 2002.

[2] M. D. Rayman, T. C. Fraschetti, C. A. Raymond, and C. T. Russell, "Coupling of system resource margins through the use of electric propulsion: Implications in preparing for the Dawn mission to Ceres and Vesta," *Acta Astronautica*, Vol. 60, No. 10-11, 2007, pp. 930 – 938.

[3] R. Jehn, "BepiColombo A mission to Mercury," *21st International Symposium on Space Flight Dynamics (ISSFD)*, Toulouse, France, Sept 2009.

[4] K. Brinkman and H. Visser, "A closed-loop guidance approximation for the turn-back maneuver after engine failure during climb-out," *45th AIAA Aerospace Sciences Meeting and Exhibit*, 2007.

[5] J. Dupacova, "Multistage Stochastic Programs: The State-of-The-Art and Selected Bibliography," *Kybernetika*, Vol. 31, No. 2, 1995, pp. 151–174.

[6] R. T. Rockafellar and R. J.-B. Wets, "Stochastic convex programming: basic duality," *Pacific J. Math*, Vol. 62, No. 1, 1976, pp. 173–195.

[7] A. Prekopa, "On Probabilistic Constrained Programming," *Princeton Symposium on Mathematical Programming* (P. U. Press, ed.), Princeton University Press, Princeton University Press, 1970, pp. 113 – 138.

[8] A. Prekopa, "Contributions to the theory of stochastic programming," *Mathematical Programming*, Vol. 4, No. 1, 1973, pp. 202 – 221.

[9] R. Bertrand and R. Epenoy, "New smoothing techniques for solving bang-bang optimal control problems-numerical results and statistical interpretation," *Optimal Control Applications and Methods*, Vol. 23, No. 4, 2002, pp. 171–197.

[10] A. Kleywegt, A. Shapiro, and T. H.-d. Mello, "The Sample Average Approximation Method for Stochastic Discrete Optimization," *SIAM Journal of Optimisation*, Vol. 12, No. 2, 2001, pp. 479 – 502.

[11] A. Greenwald, B. Guillemette, V. Naroditskiy, and M. Tschantz, "Scaling Up the Sample Average Approximation Method for Stochastic Optimization with Applications to Trading Agents," *Trading Agent Design and Analysis at the 19th International Joint Conference on Artificial Intelligence*, August 2005.

[12] S. Ahmed and A. Shapiro, "The Sample Average Approximation Method for Stochastic Programs with Integer Recourse," February 2002.

[13] A. Shapiro, D. Dentcheva, and A. Ruszczyński, *Lectures on Stochastic Programming: Modeling and Theory*, Vol. 9 of *MPS/SIAM Series on Optimization*. SIAM, 2009.

[14] H. Robbins and S. Monro, "A Stochastic Approximation Method," *Ann. Math. Stat.*, Vol. 22, 1951, pp. 400–407.

[15] H. Kushner and D. Clark, *Stochastic approximation methods for constrained and unconstrained systems*, Vol. 26 of *Applied Mathematical Sciences*. Springer-Verlag, 1 ed., 1978.

[16] M. T. Wasan, *Stochastic approximation*. Cambridge U.P., London,, 1969.

[17] J. Spall, *Introduction to Stochastic Search and Optimization: Estimation, Simulation and Control*. Hoboken, NJ: Wiley, 2003.

[18] J. Blum, "Multidimensional stochastic approximation methods," *Ann. Math. Statist*, Vol. 25, No. 1, 1954, pp. 737–744.

[19] D. Ruppert, "A Newton-Raphson Version of the Multivariate Robbins-Monro Procedure," *Ann. Statist*, Vol. 13, No. 1, 1985, pp. 236–245.

[20] J. C. Spall, "Stochastic version of second-order (Newton-Raphson) optimization using only function measurements," *WSC '95: Proceedings of the 27th conference on Winter simulation*, Washington, DC, USA, IEEE Computer Society, 1995, pp. 347–352.

[21] A. Nemirovski, A. Juditsky, G. Lan, and A. Shapiro, "Robust stochastic approximation approach to stochastic programming," *SIAM J. Optimization*, Vol. 19, 2009, pp. 1574 – 1609.

[22] M. Pelletier, "Weak convergence rates for stochastic approximation with application to multiple targets and simulated annealing," *Annals of Applied Probability*, Vol. 8, No. 1, 1998, pp. 10 – 44.

[23] L. C. W. Dixon and M. Bartholomew-Biggs, "Adjoint-control transformations for solving practical optimal control problems," *Optimal Control Applications and Methods*, Vol. 2, No. 4, 1981, pp. 365–381.