A coordinate transformation approach to indefinite materials and their perfect lenses

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Abstract

It is shown that a recently found generalization of coordinate transformations allows a geometric interpretation of many indefinite media, including perfect lenses made therefrom.We show that the perfect lens proposed by Smith and Schurig consists in a combination of time reversal and space inversion and derive alternative designs of perfect lenses made from two slabs of indefinite materials.

1. Introduction

Negative refractive index materials [1, 2] and their application to perfect lenses [3] attracted considerable interest in the recent years. The theoretical understanding of negative refraction profits a lot from the coordinate transformation approach as introduced by Pendry and Leonhardt [4, 5]. Within this picture negative refraction can be understood as space inversion [6, 7] or as time reversal inside the material. More recently, negative refraction was studied for indefinite media [8, 9], i.e. materials where the eigenvalues of ϵ and μ do not all have the same sign. Though the question of perfect lenses is addressed therein a clear and simple strategy of their design seems to be missing. This is related to the fact that indefinite materials cannot be understood by the coordinate transformations of Refs. [4, 5, 6]. Here we show how a recently found extension [10] of these transformations can overcome this restriction and how this provides an intuitive geometric picture of perfect lenses made from indefinite materials. We re-interpret the perfect lens as proposed in Ref. [8] in this framework and also provide examples of new designs.

2. A coordinate transformation approach to indefinite materials

In the coordinate transformation approach one introduces a mapping from laboratory space x^μ with metric $g_{\mu\nu}$ to electromagnetic space $\bar{x}^\mu$, $\bar{g}_{\mu\nu}$, which describes the behavior of the electromagnetic fields. In media establishing this situation permittivity and permeability are equal and proportional to the spatial part of the metric [6]: $\mu^{ij} = \epsilon^{ij} \propto \gamma^{ij}$. As the space metric by definition has three positive eigenvalues, indefinite media are not accessible with this approach. This concept can be generalized by realizing that the two sets of Maxwell's equations

$$\nabla_i B^i = 0\ , \qquad \nabla_0 B^i + \epsilon^{ijk}\partial_j E_k = 0\ , \qquad \text{and} \qquad \nabla_i D^i = \rho\ , \qquad \epsilon^{ijk}\partial_j H_k - \nabla_0 D^i = j^i\ , \tag{1}$$

have mutually excluding field content and do not depend explicitly on the media properties. Thus, two different transformations applied to the two sets still establish a symmetry of Maxwell's equations (though not of the constitutive relation) and can be used to design in a geometrically intuitive and straightforward way new materials. Assigning barred and double barred variables to the two transformations,

$$\Big(x^\mu, \vec{E}(x), \vec{B}(x)\Big) \Longrightarrow \Big(\bar{x}^\mu, \vec{\bar{E}}(\bar{x}), \vec{\bar{B}}(\bar{x})\Big)\ ; \qquad \Big(x^\mu, \vec{D}(x), \vec{H}(x)\Big) \Longrightarrow \Big(\bar{\bar{x}}^\mu, \vec{\bar{\bar{D}}}(\bar{\bar{x}}), \vec{\bar{\bar{H}}}(\bar{\bar{x}})\Big)\ , \tag{2}$$

the most general permittivity and permeability tensors of this class of media may be written as

$$\epsilon^{ij} = -\bar{s}\frac{\sqrt{-\bar{\bar{g}}}}{\sqrt{\gamma}g_{\bar{0}\bar{0}}}g^{\bar{\bar{i}}\bar{\bar{j}}} \ , \quad \mu^{ij} = \bar{s}\bar{\bar{s}}\frac{\sqrt{-\bar{g}}}{\sqrt{-\bar{\bar{g}}}}\epsilon^{ji} \ , \quad \text{with} \quad g^{\bar{\mu}\bar{\bar{\nu}}} = \bar{g}^{\mu\rho}\frac{\partial\bar{\bar{x}}^{\nu}}{\partial\bar{x}^{\rho}} = \frac{\partial\bar{x}^{\mu}}{\partial\bar{\bar{x}}^{\rho}}\bar{\bar{g}}^{\rho\nu} = \frac{\partial\bar{x}^{\mu}}{\partial x^{\rho}}g^{\rho\sigma}\frac{\partial\bar{\bar{x}}^{\nu}}{\partial x^{\sigma}} \ . \tag{3}$$

Here, $\bar{s}$ and $\bar{\bar{s}}$ denote the signs due to change of orientation in the mappings $x^\mu \to \bar{x}^\mu$ and $x^\mu \to \bar{\bar{x}}^\mu$, resp. The following facts about this result that are important: Up to an overall factor the permeability is the transposed of the permittivity, thus it is always possible to diagonalize both matrices simultaneously; the relative sign between the eigenvalues of ϵ and μ is not fixed thanks to the sign $\bar{\bar{s}}$ appearing in the equation for μ, which allows to describe media exhibiting evanescent waves; $g^{\bar{\bar{i}}\bar{\bar{j}}}$ is not a metric and no restrictions on the signs of its eigenvalues exist, which allows to design indefinite materials. The different possible permittivies can be grouped according to the signs of the eigenvalues as $(+,+,+)$, $(-,-,-)$, $(+,+,-)$ and $(-,-,+)$. Within this approach each of these four groups splits into two subgroups, depending on the relative sign of the eigenvalues of the permeability tensor compared to the permittivity. This leads exactly to the eight types of materials as they were presented in Figure 2 of Ref. [8].

3. Perfect lenses from indefinite materials

Let us assume the perfect lens of Ref. [3] as an infinite slab in the x-y plane with thickness D in the z direction. Within the approach of Ref. [6] the lens then can be understood either as a space inversion $\bar{z} = -z/\alpha$ or as a time reversal $\bar{t} = -t/\alpha$ inside the slab. A source at a distance $F < \alpha D$ from left surface of the slab is perfectly mapped onto the point at distance $\alpha D - F$ from the right surface of the slab and for $\alpha = 1$ the two prescriptions are physically equivalent.

As there exist two different coordinate transformations that establish a perfect lens the question arises, whether one can build a perfect lens by considering in (2) one mapping as a space inversion, but the other one as time reversal. Thus we assume inside the slab $\bar{z} = -z/\alpha$ and $\bar{\bar{t}} = -t/\beta$ while all other coordinates map trivially, i.e. inside the slab $\vec{E}$ and $\vec{B}$ travel backwards in space, while $\vec{D}$ and $\vec{H}$ travel backwards in time. From the geometric picture as well as from the ensuing constitutive relations with

$$\epsilon^{ij} = \text{diag}(1, 1, -1/\alpha) \ , \qquad \mu^{ij} = \text{diag}(\alpha/\beta, \alpha/\beta, -1/\beta) \ , \tag{4}$$

it is immediately seen that this does not yield a perfect lens, since the transformation connecting space inversion with time reversal represents—independently of α and β—a non-trivial transformation when applied to one pair of the fields, only. Thus we need a second slab of thickness d which exactly establishes this additional mapping. One coordinate transformation can be assumed to be trivial therein, e.g. $\bar{\bar{x}}^\mu = x^\mu$. The other one, $\bar{x}^\mu(x)$, then is a combination of time reversal and space inversion with (T and Z are unimportant additive constants)

$$\bar{t} = -\gamma t + T = -\frac{d}{D(\beta+1) - d}t + T \ , \qquad \bar{z} = -\delta z = -\frac{D(\beta+1) - d}{d + D(\alpha - \beta)}z + Z \ , \tag{5}$$

realized by a material with

$$\epsilon^{ij} = \text{diag}(-\gamma, -\gamma, \gamma\delta) \ , \qquad \mu^{ij} = \text{diag}(-1/\delta, -1/\delta, 1) \ . \tag{6}$$

This is exactly the type of perfect lens as it has been suggested in Ref. [8]. If a source at distance F from the slab shall be imaged the constraint $d < \beta D - F$ has to be met.

Of course there exist many other ways to establish perfect lenses. Here we want to mention the two possibilities as illustrated in Figure 1: again two slabs of material are assumed, in one of them exclusively $\vec{E}$ and $\vec{B}$ transform, in the other one $\vec{D}$ and $\vec{H}$. As in the standard perfect lens there exist two realizations, space inversion and time reversal. In the case of space inversion we assume for the first slab

$$\bar{z} = -\alpha z \ , \quad \bar{\bar{z}} = z \quad \Rightarrow \quad \epsilon_1^{ij} = \text{diag}(-1, -1, \alpha) \ , \qquad \mu_1^{ij} = \text{diag}(1/\alpha, 1/\alpha, -1) \ , \tag{7}$$

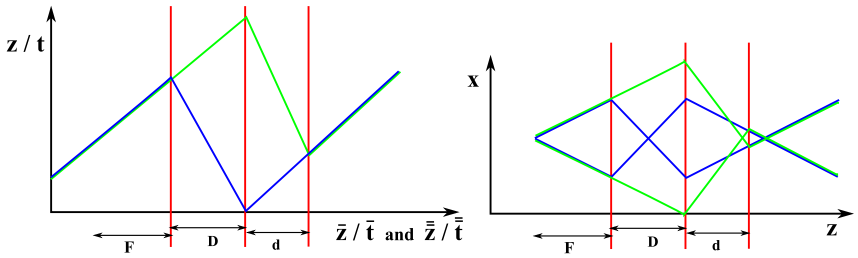

Fig. 1: Perfect lens made from two slabs of indefinite material. Left the coordinate transformation, right the imaging of a point source; the blue lines refer to the coordinates $\bar{x}^\mu$, the green ones to $\bar{\bar{x}}^\mu$.

which establishes for the second slab with $\beta = (D(\alpha+1) - d)/d$

$$\bar{z} = z \ , \quad \bar{\bar{z}} = -\beta z \quad \Rightarrow \quad \epsilon_2^{ij} = \text{diag}(1/\beta, 1/\beta, -1) \qquad \mu_2^{ij} = \text{diag}(-1, -1, \beta) \ . \tag{8}$$

Similarly, a lens based on time reversal uses the mapping $\bar{t} = -\alpha t$, requiring $\bar{\bar{t}} = -\beta t$ in the second slab, which is realized by

$$\epsilon_1^{ij} = \alpha \cdot \mathbb{1}_{3\times 3} \ , \quad \mu_1^{ij} = -\mathbb{1}_{3\times 3} \ ; \qquad \epsilon_2^{ij} = -\mathbb{1}_{3\times 3} \ , \quad \mu_2^{ij} = \beta \cdot \mathbb{1}_{3\times 3} \ . \tag{9}$$

4. Conclusion

We presented the application of a generalized formalism of coordinate transformations to indefinite media and in particular to their perfect lenses. We have shown that the eight classes of indefinite media as presented in Ref. [8] allow a geometric interpretation and that the perfect lens suggested there is a combination of time reversal and space inversion applied differently to the two sets of fields. Many more possibilities to create a perfect lens exist, two of them have been presented here. Still, there exist additional constraints not discussed in this paper. In particular, impedance matching with the vacuum is more subtle in the result (3) than in the standard coordinate transformation, which in practice will limit the usability of many designs. We note that this is related to the fact that the Poynting vector does not transform covariantly under the transformation (2).

References

[1] V. Veselago, Electrodynamics of substances with simultaneously negative values of sigma and mu, *Sov. Phys. Usp.* vol. 10, p. 509, 1968.
[2] V. Veselago, L. Braginsky, V. Shkover, and C. Hafner, Negative refractive index materials, *J. Comput. Theor. Nanosci* vol. 3,p. 1, 2006.
[3] J. B. Pendry, Negative refraction makes a perfect lens, *Phys. Rev. Lett.* vol. 85, p. 3966, 2000.
[4] J. Pendry, D. Schurig, and D. Smith, Controlling electromagnetic fields, *Science* vol. 312, p.1780, 2006.
[5] U. Leonhardt, Optical conformal mapping, *Science* vol. 312, p.1777, 2006.
[6] U. Leonhardt and T. G. Philbin, General relativity in electrical engineering, *New Journal of Physics* vol. 8, p. 247, 2006.
[7] D. Schurig, J. Pendry, and D. Smith, Transformation-designed optical elements, *Opt. Express* vol. 15, p. 14772, 2007.
[8] D. Smith and D. Schurig, Electromagnetic wave propagation in media with indefinite permeability and permittivity tensors, *Phys. Rev. Lett.* vol. 90, p. 077405, 2003.
[9] D. Smith, P. Kolilnko, and D. Schurig, Negative refraction in indefinite media, *J. Opt. Soc. Am. B* vol. 21, p.1032, 2004.
[10] L. Bergamin, Triple-spacetime metamaterials, paper submitted to Metamaterials' 2008.