Triple-spacetime Metamaterials

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Abstract

In this paper an extension of the coordinate transformation approach to artificial media as introduced by Pendry and Leonhardt is presented. It is based upon the fact that two different transformations acting on $(\vec{E}, \vec{B})$ and $(\vec{D}, \vec{H})$, resp., establish a symmetry of Maxwell's equations, but change the constitutive relation. This allows a geometric interpretation of non-reciprocal (ϵ and μ not symmetric) and so-called indefinite media.

1. Introduction

The concept of Metamaterials mimicking coordinate transformations [1, 2, 3] has been very successful to design and understand novel types of media, such as invisibility cloaks [1, 2], perfect lenses, artificial black holes [3] or magnification devices [4]. Nevertheless, the class of constitutive relations available in this way is rather limited as the medium must be reciprocal and have equal permittivity and permeability, a complementary approach not making use of symmetries and geometric interpretations has been proposed in [5]. In this paper we provide a generalization of [3], which in terms of accessible media is less general than [5], but allows an immediate geometric interpretation and is based on symmetry transformations.

2. Triple-spacetime Metamaterials

The constitutive relations of electromagnetism in vacuo resemble those of a reciprocal medium if Maxwell's equations are written in terms of an arbitrary (not necessarily flat) metric $g_{\mu\nu}$ [6]. Therefore, if empty space can look as a medium, a medium could also look as empty space. This is indeed possible [3]: one starts from laboratory space x^μ with metric $g_{\mu\nu}$ and applies a (eventually singular) symmetry transformation (diffeomorphism), which locally is expressed by a coordinate transformation $x^\mu \to \bar{x}^\mu$. The new space with metric $\bar{g}_{\mu\nu}$ describes the behavior of the electromagnetic fields, but the equations are re-interpreted in terms of the original coordinates x^μ. This leads to the constitutive relations [3]

$$\tilde{D}^i = \pm \frac{\bar{g}^{ij}}{\sqrt{-\bar{g}_{00}}} \frac{\sqrt{\bar{\gamma}}}{\sqrt{\gamma}} \tilde{E}_j - \frac{\bar{g}_{0j}}{\bar{g}_{00}} \epsilon^{jil} \tilde{H}_l \ , \qquad \tilde{B}^i = \pm \frac{\bar{g}^{ij}}{\sqrt{-\bar{g}_{00}}} \frac{\sqrt{\bar{\gamma}}}{\sqrt{\gamma}} \tilde{H}_j + \frac{\bar{g}_{0j}}{\bar{g}_{00}} \epsilon^{jil} \tilde{E}_l \ , \tag{1}$$

where the sign ambiguity refers to the fact that the transformation can change the orientation of the frame. One can extend this approach by realizing that the equations of motion may exhibit invariant transformations that are not symmetries of the action. Indeed, Maxwell's equations divide into two sets of equations,

$$\nabla_i B^i = 0 \ , \quad \nabla_0 B^i + \epsilon^{ijk} \partial_j E_k = 0 \ , \quad \text{and} \quad \nabla_i D^i = \rho \ , \quad \epsilon^{ijk} \partial_j H_k - \nabla_0 D^i = j^i \ , \tag{2}$$

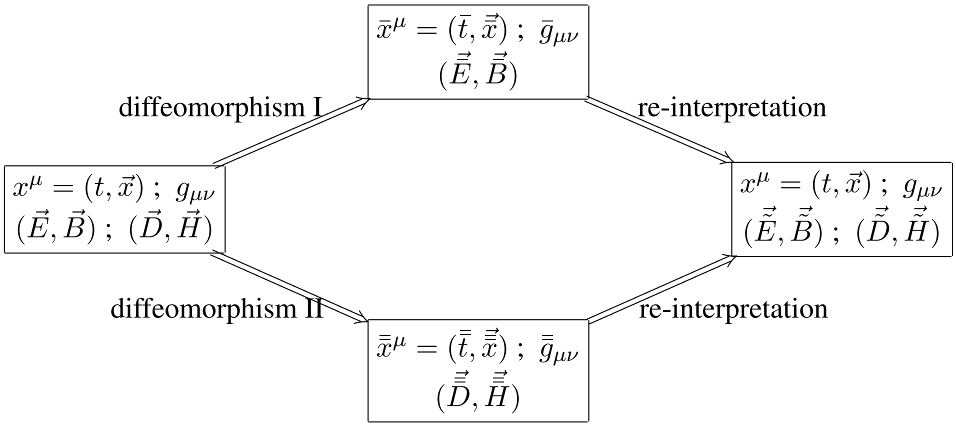

Fig. 1: Illustration and notation of the generalized "triple spacetime Metamaterials". Notice that the diffeomorphism I only acts on the fields $\vec{E}$ and $\vec{B}$, while diffeomorphism II acts on $\vec{D}$ and $\vec{H}$.

which have mutually excluding field content and do not depend explicitly on the media properties. Therefore, two different diffeomorphisms applied to the two sets leave Maxwell's equations invariant, i.e. any source-free solution thereof is mapped upon another source-free solution. Still, the action as well as the constitutive relation are not invariant and thus the two situations are physically inequivalent. Nonetheless, this deformation allows an immediate geometric interpretation, as it simply states that we can deform independently from each others the spaces for $(\vec{E}, \vec{B})$ and $(\vec{D}, \vec{H})$ (cf. Fig. 1.)

To arrive at the constitutive relations we proceed analogously to Ref. [3]: all equations are rewritten in terms of the laboratory metric $g_{\mu\nu}$, whereby the field strength tensor $\bar{F}_{\mu\nu} = (\vec{\bar{E}}, \vec{\bar{B}})$, the excitation tensor $\bar{\bar{\mathcal{H}}}^{\mu\nu} = (\vec{\bar{\bar{D}}}, \vec{\bar{\bar{H}}}))$ and the four-current $\bar{\bar{J}}^\mu = (\bar{\bar{\rho}}, \vec{\bar{\bar{j}}})$ must be rescaled as

$$\tilde{F}_{\mu\nu} = \pm \bar{F}_{\mu\nu}\ , \qquad \tilde{\mathcal{H}}^{\mu\nu} = \frac{\sqrt{-\bar{\bar{g}}}}{\sqrt{-g}} \bar{\bar{\mathcal{H}}}^{\mu\nu}\ , \qquad \tilde{J}^\mu = \frac{\sqrt{-\bar{\bar{g}}}}{\sqrt{-g}} \bar{\bar{J}}^\mu\ . \tag{3}$$

From this, the relativistically covariant formulation of the constitutive relation follows immediately. After rewriting everything in terms of space vectors one finds

$$\tilde{D}^i = -\bar{s} \frac{\sqrt{-\bar{\bar{g}}}}{\sqrt{\gamma} g_{\bar{0}\bar{\bar{0}}}} g^{\bar{\bar{i}}\bar{j}} \tilde{E}_j - \bar{s}\bar{\bar{s}} \frac{\sqrt{-\bar{g}}\sqrt{-\bar{\bar{g}}}}{\gamma g_{\bar{0}\bar{\bar{0}}}} g^{\bar{\bar{i}}\bar{k}} g^{\bar{\bar{0}}\bar{l}} \epsilon_{klm} g^{\bar{m}\bar{\bar{j}}} \tilde{H}_j\ , \tag{4}$$

$$\tilde{B}^i = -\bar{\bar{s}} \frac{\sqrt{-\bar{g}}}{\sqrt{\gamma} g_{\bar{0}\bar{\bar{0}}}} g^{\bar{i}\bar{\bar{j}}} \tilde{H}_j + \bar{s}\bar{\bar{s}} \frac{\sqrt{-\bar{g}}\sqrt{-\bar{\bar{g}}}}{\gamma g_{\bar{0}\bar{\bar{0}}}} g^{\bar{i}\bar{\bar{k}}} \epsilon_{klm} g^{\bar{\bar{l}}\bar{0}} g^{\bar{\bar{m}}\bar{j}} \tilde{E}_j\ , \tag{5}$$

where $\bar{s}$ and $\bar{\bar{s}}$ are the respective signs from possible changes of orientation in the mappings $x^\mu \to \bar{x}^\mu$ and $x^\mu \to \bar{\bar{x}}^\mu$ and the symbol $g^{\bar{\mu}\bar{\bar{\nu}}}$ is defined as

$$g^{\bar{\bar{\mu}}\bar{\nu}} = \frac{\partial \bar{\bar{x}}^\mu}{\partial x^\rho} \frac{\partial \bar{x}^\nu}{\partial x^\sigma} g^{\rho\sigma} = \bar{\bar{g}}^{\mu\rho} \frac{\partial \bar{x}^\nu}{\partial \bar{\bar{x}}^\rho} = \frac{\partial \bar{\bar{x}}^\mu}{\partial \bar{x}^\rho} \bar{g}^{\rho\nu}\ . \tag{6}$$

The result (4) and (5) reduces to the relations (1) if $g_{\bar{\mu}\bar{\bar{\nu}}}$ is a symmetric matrix of signature $(3,1)$. This does not imply $\bar{x}^\mu = \bar{\bar{x}}^\mu$ but rather that there exists yet a different space which describes the same media properties in terms of a single transformation. Our generalization exhibits the following features:

- As $g^{\bar{\bar{i}}\bar{j}} = (g^{\bar{j}\bar{\bar{i}}})^T$ it follows that permittivity and permeability are related as $\bar{\bar{s}}\sqrt{-\bar{\bar{g}}}\mu^{ij} = \bar{s}\sqrt{-\bar{g}}\epsilon^{ji}$. It should not come as a surprise that permittivity and permeability cannot be independent as by virtue of the definition of the relativistically covariant tensors $F_{\mu\nu}$ and $\mathcal{H}_{\mu\nu}$ no invariant transformation can act independently on $\vec{E}$ and $\vec{B}$ or $\vec{D}$ and $\vec{H}$, resp.

- Permittivity and permeability need no longer be symmetric. Therefore it is possible to describe non-reciprocal materials. This happens if the mapping between the two electromagnetic spaces, $\partial\bar{x}^\mu/\partial\bar{\bar{x}}^\nu$, is not symmetric in μ and ν, e.g. for a material with mapping $\bar{x} = x - z$, $\bar{\bar{x}} = x + z$.

- The generalized transformations yield many more possibilities considering the signs of the eigenvalues of permittivity and permeability. Within the method of Ref. [3], μ and ϵ are determined by the spatial metric of the deformed space, which by definition must have three positive eigenvalues.

  Within the generalized setup of "triple space Metamaterials", however, the signs of the eigenvalues in ϵ (or μ) can be chosen freely as no restrictions of this type exist for the transformation matrices. Such indefinite media [7, 8] emerge if certain space directions are inverted differently in the two mappings, e.g. $\bar{x} = -x$, $\bar{\bar{x}} = x$. Furthermore the relative sign between the eigenvalues of ϵ and those of μ can be chosen as is seen from (4) and (5). This allows media exhibiting evanescent waves as a consequence of different time directions, $\bar{t} = -t$ but $\bar{\bar{t}} = t$. We note that all eight classes of materials discussed in Ref. [7] allow a geometric interpretation within the setup of "triple spacetime Metamaterials."

- More complicated than permittivity and permeability are the bi-anisotropic couplings. From

$$\xi^{ij} = -\bar{s}\bar{\bar{s}}\frac{\sqrt{-\bar{g}}\sqrt{-\bar{\bar{g}}}}{\gamma g_{\bar{0}\bar{\bar{0}}}} g^{\bar{i}\bar{\bar{k}}} g^{\bar{\bar{0}}\bar{l}} \epsilon_{klm} g^{\bar{m}\bar{\bar{j}}} \,, \qquad \kappa^{ij} = \bar{s}\bar{\bar{s}}\frac{\sqrt{-\bar{g}}\sqrt{-\bar{\bar{g}}}}{\gamma g_{\bar{0}\bar{\bar{0}}}} g^{\bar{i}\bar{\bar{k}}} \epsilon_{klm} g^{\bar{l}\bar{\bar{0}}} g^{\bar{\bar{m}}\bar{j}} \,, \tag{7}$$

  it follows similarly to Eq. (1) that all electric-magnetic couplings vanish if $x^0 = \bar{x}^0 = \bar{\bar{x}}^0$. Thus again we find that electric-magnetic couplings are the result of non-trivial transformations of time.

4. Conclusion

We have introduced in this paper a generalization of the concept of coordinate transformations to design artificial materials. Similar to the standard coordinate transformations this technique offers a geometric interpretation, but in contrast to the former it allows to design a wider class of media, including non-reciprocal media, indefinite media and materials exhibiting evanescent waves. We mention that the most general class of media that can be obtained from invariant transformations of the equations of motion is even more general than the class presented here: indeed, a continuous version of electric-magnetic duality leaves Eqs. (2) invariant, however a geometric interpretation thereof as well as its application to media including sources are less immediate. Besides practical applications also theoretical questions remain open, such as impedance matching with the vacuum and the behavior of conservation equations (in particular the Poynting vector) under the suggested transformations.

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